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ON THE CLASS OF (A, n)-REAL POWER POSITIVE OPERATORS IN SEMI-HILBERTIAN SPACE

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ABSTRACT

In this paper, the concept of the class of n-Real power positive operators on a hilbert space defined by Abdelkader Benali in [1] is generalized when an additional semi-inner product is considered. This new concept is described by means of oblique projections. For a Hilbert space operator $T \in B(H)$ is (A, n)-Real power positive operators for some positive operator A and for some positive integer n if

$$T^n + T^{\sharp n} \geq_A 0, n = 1,2, \dots.$$

KEYWORDS: Real power, Semi-Hilbertian space, Semi-inner product, Positive operators. **2000 Mathematics Subject Classification: Primary 47B20. Secondary 47B99.**

1 INTRODUCTION

A bounded linear operator T on a complex Hilbert space is n-Real power positive operators if $T^n + T^{\sharp n} \geq 0$. The class of (A, n) –power positive operators was introduced and studied by Sidi Hamidou Jah see [16], from the definition, it is easily seen that this class contains power positive operators, in [12] the authors O.A. Mahmoud Sid Ahmed introduced the class $n-$ power quasi normal operators and study some properties of such class for different values of the parameter n. In [1] we introduce a new class of operators T namely n-real power positive operator denoted by $[n\mathcal{RP}]$satisfying $T^n + T^{*n} \geq 0$, for $n = 1,2,3, \dots$.

The purpose of this paper is to study the class of (A, n)-Real power positive operators in semi-hilbertian spaces, denoted by $[n\mathcal{RP}]_A$.

2 (A, n)-REAL POWER POSITIVE OPERATORS

Definition 2.1 *For $n \in \mathbb{N}$, an operator $T \in \mathcal{L}(H)$ is said to be (A, n)-real power positive operator if $T^n + T^{\sharp n} \geq_A 0$ or equivalenty $A(T^n + T^{\sharp n}) \geq 0$.*

Proposition 2.1 *Let $T \in \mathcal{B}_A(\mathcal{H})$ and $n \in \mathbb{N}$ the following properties hold*

(1) if $T \in [n\mathcal{RP}]_A$ then so $T^{\sharp}$.
(2) $T \in [n\mathcal{RP}]_A$ if and only if $Re\langle T^n x \mid x\rangle \geq_A 0 \quad \forall \quad x \in \mathcal{H}$.
(3) If T is invertible then $T \in [n\mathcal{RP}]_A$ if and only if $T^{-1} \in [n\mathcal{RP}]_A$.

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Proof. (1) Obvious from the definition 2.1.
(2) In fact, it is well known that

$$\begin{aligned} T \in [n\mathcal{RP}]_A &\Leftrightarrow T^n + T^{\sharp n} \geq_A 0 \Leftrightarrow \left\langle (T^n + T^{\sharp n})x \mid x \right\rangle_A \geq 0 \quad \forall \quad x \in \mathcal{H} \\ &\Leftrightarrow \langle T^n x \mid x \rangle_A + \langle T^{*n} x \mid x \rangle_A \geq 0 \quad \forall \quad x \in \mathcal{H} \\ &\Leftrightarrow \langle T^n x \mid x \rangle_A + \langle x \mid T^n x \rangle_A \geq 0 \quad \forall \quad x \in \mathcal{H} \\ &\Leftrightarrow \langle T^n x \mid x \rangle_A + \overline{\langle T^n x \mid x \rangle}_A \geq 0 \quad \forall \quad x \in \mathcal{H} \\ &\Leftrightarrow 2Re\langle T^n x \mid x \rangle_A \geq_A 0. \end{aligned}$$

(3)Assume that T is invertible and $T \in [n\mathcal{RP}]_A$ we have $Re\langle T^n x \mid x \rangle \geq_A 0 \quad \forall \quad x \in \mathcal{H}$. It follows that for all $x \in \mathcal{H}$,

$$0 \leq Re\langle T^n T^{-n} x \mid T^{-n} x \rangle_A = Re\langle x \mid T^{-n} x \rangle_A = Re\overline{\langle T^{-n} x \mid x \rangle}_A = Re\langle T^{-n} x \mid x \rangle_A.$$

Hence $T^{-1} \in [n\mathcal{RP}]_A$. The converse is obvious.
The following examples show that the two classes $[n\mathcal{RP}]_A$ and $[(n+1)\mathcal{RP}]_A$ are not the same.

Example 2.1 *Let* $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^2)$. *A simple computation shows that*

$$T^\sharp = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad and \quad T^n + T^{\sharp n} = n\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For all $(u, v) \in \mathbb{C}^2$ we have

$$\left\langle (T^n + T^{\sharp n}) \begin{pmatrix} u \\ v \end{pmatrix} \mid \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_A = 0 \geq_A 0.$$

So $T \in [n\mathcal{RP}]_A$.

Example 2.2 *Let* $T = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^3)$. *It is easy to see that* $T \notin [n\mathcal{RP}]_A$ *for all* $n = 1, 2, \ldots$.

Example 2.3 *Let* $T = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^2)$. *A simple computation shows that*

$$T^\sharp = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, T^2 + T^{\sharp 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad and \quad T^3 + T^{\sharp 3} = 4\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For all $(u, v) \in \mathbb{C}^2$ we have

$$\left\langle (T^2 + T^{\sharp 2}) \begin{pmatrix} u \\ v \end{pmatrix} \mid \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_A = 0 \geq_A 0.$$

Hence $T \in [2\mathcal{RP}]_A$.

On the other hand

$$\left\langle (T^3 + T^{\sharp 3}) \begin{pmatrix} u \\ v \end{pmatrix} \mid \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_A = -u^2 - v^2 \leq_A 0.$$

So $T \notin [3\mathcal{RP}]_A$.

Proposition 2.2 *If S, $T \in B_A(H)$ are unitarily equivalent and if T is (A-n)-real power positive operators then so is S.*

Proof. Let T be an (A-n)-real power positive operator and S be unitary equivalent of T. Then there exists unitary operator U such that $S = UTU^\sharp$ so $S^n = UT^nU^\sharp$
We have

$$T \in [n\mathcal{RP}]_A \Leftrightarrow T^n + T^{\sharp n} \geq_A 0 \Leftrightarrow \left(T^n + T^{\sharp n}\right)U^\sharp \geq_A 0$$
$$\Leftrightarrow U\left(T^n + T^{\sharp n}\right)U^\sharp \geq_A 0$$
$$\Leftrightarrow UT^nU^\sharp + UT^{\sharp n}U^\sharp \geq_A 0$$
$$\Leftrightarrow S^n + S^{\sharp n} \geq_A 0$$
$$\Leftrightarrow S \in [n\mathcal{RP}]_A$$

Theorem 2.1 *Let $T, S \in [n\mathcal{RP}]_A$ such that $T^kS = -S^kT$ for $k = 1,2,\ldots,n-1$ with $n \geq 2$, then $T + S \in [n\mathcal{RP}]_A$.*
Proof. From the hypothesis it is clear that $(T+S)^n = T^n + S^n$ and so that

$$(T+S)^n + (T^\sharp + S^\sharp)^n = \underbrace{T^n + T^{\sharp n}}_{\geq_A 0} + \underbrace{S^n + S^{\sharp n}}_{\geq_A 0} \geq_A 0.$$

Lemma 2.1 *Let $T, S \in \mathcal{B}(\mathcal{H})$ such that $T \geq_A S$ and let $R \in \mathcal{B}_A(\mathcal{H})$. Then the following properties hold*
(1) $R^\sharp TR \geq_A R^\sharp SR$.
(2) $RTR^\sharp \geq_A RSR^\sharp$.
(3) If R is A-selfadjoint then $RTR \geq_A RSR$.

Proposition 2.3 *If $T \in [n\mathcal{RP}]_A$ is such that $T^\sharp T^2 = T^2T^\sharp$ then $T^\sharp T^2 \in [n\mathcal{RP}]_A$.*
Proof. Since $T \in [n\mathcal{RP}]_A$ we have by Lemma 3.1 that

$$T^n + T^{\sharp n} \geq_A 0 \Longrightarrow T^{\sharp n}T^nT^n + T^{\sharp 2n}T^n \geq_A 0$$
$$\Longrightarrow (T^\sharp T^2)^n + (T^{\sharp 2}T)^n \geq_A 0 (\ since \quad T^\sharp T^2 = T^2T^\sharp)$$
$$\Longrightarrow (T^\sharp T^2)^n + (T^\sharp T^2)^{\sharp n} \geq_A 0.$$

Hence $T^\sharp T^2 \in [n\mathcal{RP}]_A$ as required.

Proposition 2.4 *Let $T \in \mathcal{B}_A(\mathcal{H})$. Consider $F = T^{n-1} + T^\sharp$ and $G = T^{n-1} - T^\sharp$ for $n \in \mathbb{N}$. If T is normal then the following equivalence holds*

$$T \in [n\mathcal{RP}]_A \quad if\ and\ only\ if \quad FF^\sharp \geq_A GG^\sharp.$$

Proof. Since T is normal we have

$$FF^\sharp - GG^\sharp = (T^{n-1} + T^\sharp)(T^{\sharp n-1} + T) - (T^{n-1} - T^\sharp)(T^{\sharp n-1} - T)$$
$$= T^n + T^{\sharp n}.$$

From which it follows that

$$T \in [n\mathcal{RP}]_A \Leftrightarrow T^n + T^{\sharp n} \geq_A 0 \Leftrightarrow FF^\sharp - GG^\sharp \geq_A 0.$$

Proposition 2.5 *Let $T \in \mathcal{B}_A(\mathcal{H})$.*
(1) If T is almost subprojection, then

$$T \in [2\mathcal{RP}]_A \quad if\ and\ only\ if \quad T \in [4\mathcal{RP}]_A.$$

(2) If T is idempotent, then

$$T \in [\mathcal{RP}]_A \quad if\ and\ only\ if \quad T \in [n\mathcal{RP}]_A.$$

Proof. (1) T is almost subprojection, $T^4 = T^{\sharp 2}$ for all $x \in \mathcal{H}$ (see [4]) we have

$$Re\langle T^2x \mid x\rangle_A = Re\left\langle T^{\sharp 4}x \mid x\right\rangle_A = Re\langle x \mid T^4x\rangle_A = Re\overline{\langle T^4x \mid x\rangle}_A = Re\langle T^4x \mid x\rangle_A$$

So

$$T \in [2\mathcal{RP}]_A \geq 0 \Leftrightarrow T \in [4\mathcal{RP}]_A \geq 0.$$

(2) Since T is idempotent we have $T = T^2 = \ldots = T^n$ and so that

$$T^n + T^{\sharp n} = T + T^\sharp.$$

Hence the desired result.

The following examples show that an operator $T \in [n\mathcal{RP}]_A$ need not be almost subprojection and vice versa.

Example 2.4 *Let* $\square = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ $\square\square\square$ $\square = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *be an operator acting in two- dimensional complex Hilbert space. then* $\square \in [\square\square\square]_{\square}$ *for all* $\square \in \square$*. Now, by direct calculation* $\square^4 = \begin{pmatrix} 1 & 0 \\ 0 & 16 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = \square^{\sharp 2}$

Theorem 2.2 *Let* $\square, \square \in \square_{\square}(\square)$ *such that* $\square\square = \square\square = \square + \square$*. If* $\square$ *and* $\square$ *are in* $[\square\square\square]_{\square}$ *for* $\square = 1,2,\dots,\square$ *,then* $\square\square \in [\square\square\square]_{\square}$*.*

Proof. For $\square = 1$. Assume that $\square$ and $\square$ are in $[\square\square]_{\square}$. We have

$$\square\square + (\square\square)^{\sharp} = \square + \square^{\sharp} + \square + \square^{\sharp} \geq_{\square} 0$$

and so $\square\square \in [\square\square]_{\square}$.

For $\square = 2$. Assume that $\square$ and $\square$ are in $[\square\square\square]_{\square}$ for $\square = 1,2$. We have

$$\begin{aligned}(\square\square)^2 + (\square\square)^{\sharp 2} &= (\square + \square)^2 + (\square^{\sharp} + \square^{\sharp})^2 \\ &= \square^2 + 2\square\square + \square^2 + \square^{\sharp 2} + 2\square^{\sharp}\square^{\sharp} + \square^{\sharp 2} \\ &= \underbrace{\square^2 + \square^{\sharp 2}}_{\geq_{\square} 0} + 2\underbrace{(\square\square + (\square\square)^{\sharp})}_{\geq_{\square} 0} + \underbrace{\square^2 + \square^{\sharp 2}}_{\geq_{\square} 0}\end{aligned}$$

and so $\square\square \in [2\square\square]_{\square}$. Assume that this result is true for $\square - 1$ and we prove it for $\square$. Let $\square$ and $\square$ be in $[\square\square\square]_{\square}$ for $\square = 1,2,\dots,\square$. Since $\square\square = \square\square = \square + \square$ we have $(\square\square)^{\square} + (\square\square)^{*\square} = (\square + \square)^{\square} + (\square^{\sharp} + \square^{\sharp})^{\square}$

$$= \square^{\square} + \square^{\sharp\square} + \sum_{1 \leq \square \leq \square - 1} \binom{\square}{\square} (\square^{\square}\square^{\square - \square} + \square^{\sharp\square}\square^{\sharp\square - \square}) + \square^{\square} + \square^{\sharp\square}.$$

It suffice to prove under the assumptions that $\square^{\square}\square^{\square - \square} + \square^{\sharp\square}\square^{\sharp\square - \square} \geq_{\square} 0$,for $\square = 1,2,\dots,\square - 1$.

For $\square = 1$ we have

$$\begin{aligned}\square\square^{\square - 1} + \square^{\sharp}\square^{\sharp\square - 1} &= \square\square\square^{\square - 2} + \square^{\sharp}\square^{\sharp}\square^{\sharp\square - 2} \\ &= (\square + \square)\square^{\square - 2} + (\square^{\sharp} + \square^{\sharp})\square^{\sharp\square - 2} \\ &= \square\square^{\square - 2} + \square^{\sharp}\square^{\sharp\square - 2} + \underbrace{\square^{\square - 1} + \square^{\sharp\square - 1}}_{\geq_{\square} 0} \\ &= \square\square\square^{\square - 3} + \square^{\sharp}\square^{\sharp}\square^{\sharp\square - 3} + \underbrace{\square^{\square - 1} + \square^{\sharp\square - 1}}_{\geq_{\square} 0} \\ &= \square\square^{\square - 3} + \square^{\sharp}\square^{\sharp\square - 3} + \underbrace{\square^{\square - 2} + \square^{\sharp\square - 2}}_{\geq_{\square} 0} + \underbrace{\square^{\square - 1} + \square^{\sharp\square - 1}}_{\geq_{\square} 0} \\ &= \cdots \quad \dots \quad \dots \\ &= \sum_{1 \leq \square \leq \square - 1} (\underbrace{\square^{\square} + \square^{\sharp\square}}_{\geq_{\square} 0} + \underbrace{\square^{\square} + \square^{\sharp\square}}_{\geq_{\square} 0}).\end{aligned}$$

For $\square = 2$ we have

$$\begin{aligned}\square^2\square^{\square - 2} + \square^{\sharp 2}\square^{\sharp\square - 2} &= \square\square\square\square^{\square - 3} + \square^{\sharp}\square^{\sharp}\square^{\sharp}\square^{\sharp\square - 3} \\ &= \square^2\square^{\square - 3} + \square\square^{\square - 2} + \square^{\sharp 2}\square^{*\square - 3} + \square^{\sharp}\square^{\sharp\square - 2} \\ &= \square^2\square^{\square - 4} + \square\square^{\square - 3} + \square\square^{\square - 2} + \square^{\sharp 2}\square^{\sharp\square - 4} + \square^{\sharp}\square^{\sharp\square - 3} + \square^{\sharp}\square^{\sharp\square - 2} \\ &= \square^2\square^{\square - 5} + \square\square^{\square - 4} + \square\square^{\square - 3} + \square\square^{\square - 2} \\ &\quad + \square^{\sharp 2}\square^{\sharp\square - 5} + \square^{\sharp}\square^{\sharp\square - 4} + \square^{\sharp}\square^{\sharp\square - 3} + \square^{\sharp}\square^{\sharp\square - 2} \\ &= \cdots \quad \dots \quad \dots \quad \dots \\ &= (\square\square)^2 + (\square\square)^{\sharp 2} + \sum_{1 \leq \square \leq \square - 2} (\square\square^{\square} + T^{\sharp}S^{\sharp k}).\end{aligned}$$

A simple calculation shows that

$$TS^k + T^{\sharp}S^{\sharp k} = T + T^{\sharp} + \sum_{1 \leq j \leq k} (S^j + S^{\sharp j}).$$

We deduce that $T^2S^{n-2} + T^{\sharp 2}S^{\sharp n-2}$

$$= (TS)^2 + (TS)^{\sharp 2} + \sum_{1 \leq k \leq n-2} (T + T^{\sharp} + \sum_{1 \leq j \leq k} (S^j + S^{\sharp j})) \geq_A 0.$$

Same way for $p = 3, \ldots, n-1$. Hence $(TS)^n + (TS)^{\sharp n} \geq_A 0$ as required.

Example 2.5 *Let* $S = T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. *it is easy to see that* $T \in [k\mathcal{RP}]_A$ *for* $k = 1,2,\ldots,n$ *and* $TS \in [n\mathcal{RP}]_A$.

The following example shows that Theorem 2.3 is not necessarily true if □□ ≠ □ + □.

Example 2.6 *Let* □ = $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, □ = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *and* □ = $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. *We have* □ *and* □ *in* [ℛ□]$_□$, □□ ≠ □ + □ *and* □□ ∉ [2ℛ□]$_□$.

Proposition 2.6 *Let* □, □ ∈ $\mathcal{B}_□(\mathcal{H})$. *If* □ ∈ [□ℛ□]$_□$ *and* □ *is unitary equivalent to* □, *then* □ ∈ [□ℛ□]$_□$.

Proof. By assumption, there is a unitary equivalent operator $\mathrm{U} \in \mathcal{B}_{\mathrm{A}}(\mathcal{H})$ such that $\mathrm{S} = \mathrm{U}^{-1}\mathrm{TU}$, which implies that

□$^\sharp$ = □$^\sharp$□$^\sharp$(□$^{-1}$)$^\sharp$ = □$^\sharp$□$^\sharp$(□$^\sharp$)$^{-1}$.

Thus we have

□$^□$ = □$^{-1}$□□□$^{-1}$□□… □$^{-1}$□□ = □$^{-1}$□$^□$□

−□$^{\sharp□}$ = −□$^\sharp$□$^\sharp$(□$^\sharp$)$^{-1}$… □$^\sharp$□$^\sharp$(□$^\sharp$)$^{-1}$
= −□$^\sharp$□$^{\sharp□}$(□$^\sharp$)$^{-1}$.

Since □ is A-unitary and using the fact that □$^□$ ≥$_□$ − □$^{\sharp□}$ we conclude that

□$^{-1}$□$^□$□ ≥$_□$ − $U^\sharp$□$^{\sharp□}$(□$^\sharp$)$^{-1}$.

Thus □$^□$ ≥$_□$ − □$^{\sharp□}$.

Theorem 2.3 *Let* □ ∈ □$_□$(□) *the following properties hold*

(1) If □$^□$ is unitary equivalent to □$^{\sharp□-1}$ then
□ ∈ [□□□]$_□$ ⇔ □ ∈ [(□ − 1)□□]$_□$, □ = 2,3,….

(2) If □$^□$ is unitary equivalent to □$^{\sharp□-1}$ for □ = 1.….□ then
□ ∈ [□□□]$_□$ ⇔ □ ∈ [□□]$_□$, □ = 2,3,….

Proof. (1) From the hypothesis there exists an operator $\mathrm{U} \in \mathcal{B}_{\mathrm{A}}(\mathcal{H}) : \mathrm{U}^\sharp\mathrm{U} = \mathrm{UU}^\sharp = \mathrm{P}_{\overline{\mathcal{R}(\mathrm{A})}}$ such that T^{n} = □$^\sharp$□$^{\sharp□-1}$□.

Firstly, assume that □ ∈ [□□□]$_□$, it follows that

□$^□$ + □$^{\sharp□}$ ≥$_□$ 0 ⟹ □$^\sharp$□$^{\sharp□-1}$□ + □$^\sharp$□$^{□-1}$□ ≥$_□$ 0 ⟹ □$^\sharp$(□$^{□-1}$ + □$^{\sharp□-1}$)□ ≥$_□$ 0.

By Lemma 2.1, we deduce that □$^{□-1}$ + □$^{\sharp□-1}$ ≥$_□$ 0 and hence □ ∈ [(□ − 1)□□]$_□$.

Conversely, assume that □ ∈ [(□ − 1)□□]$_□$. We have by Lemma 2.1

□$^{□-1}$ + □$^{\sharp□-1}$ ≥$_□$ 0 ⟹ □$^\sharp$(□$^{□-1}$ + □$^{\sharp□-1}$)□ ≥$_□$ 0 ⟹ □$^□$ + □$^{\sharp□}$ ≥$_□$ 0.

Hence □ ∈ [□□□]$_□$.

(2) From the hypothesis we have

□$^□$ = □$^\sharp_□$□$^{\sharp□-1}$□$_□$ □□□ □ = 1,2,….,□.

If we assume that □ ∈ [□□□]$_□$ we have from (1) that □ ∈ [(□ − 1)□□]$_□$. Repeating the process with □ ∈ [(□ − 1)□□]$_□$ we obtain that □ ∈ [(□ − 2)□□]$_□$. Hence the following implications hold

□ ∈ [□□□]$_□$ ⟹ □ ∈ [(□ − 1)□□]$_□$ ⟹ □ ∈ [(□ − 2)□□]$_□$ ⟹… □ ∈ [2□□]$_□$ ⟹ □ ∈ [□□]$_□$.

Conversely, assume that □ ∈ [□□]. By Lemma 2.1 we obtain

□^{2} + □$^{\sharp 2}$ = □$^\sharp_2$(□ + □$^\sharp$)□$_2$ ≥$_□$ 0 ⟹ □ ∈ [2□□]$_□$.

Also

□^{3} + □$^{\sharp 3}$ = □$^\sharp_3$(□^{2} + □$^{\sharp 2}$)□$_3$ ≥$_□$ 0 ⟹ □ ∈ [3□□]$_□$.

Repeating the process we obtain

$$\square^{\square} + \square^{\sharp\square} = \square^{\sharp}_{\square}(\square^{\square-1} + \square^{\sharp\square-1})\square_{\square} \geq_{\square} 0 \Longrightarrow \square \in [\square\square\square]_{\square}.$$

This completes the proof.

Proposition 2.7 *If* $\square \in [\square\square\square]_{\square}$ *is such that* $\square^{\sharp}\square^2 = \square^2\square^{\sharp}$ *then* $\square^{\sharp}\square^2 \in [\square\square\square]_{\square}$.

Proof. Since $T \in [n\mathcal{RP}]_A$ we have by Lemma 2.1 that

$$\square^{\square} + \square^{\sharp\square} \geq_{\square} 0 \Longrightarrow \square^{\sharp\square}\square^{\square}\square^{\square} + \square^{\sharp 2\square}\square^{\square} \geq_{\square} 0$$

$$\Longrightarrow (\square^{\sharp}\square^2)^{\square} + (\square^{\sharp 2}\square)^{\square} \geq_{\square} 0(\;\square\square\square\square\square\quad \square^{\sharp}\square^2 = \square^2\square^{\sharp})$$

$$\Longrightarrow (\square^{\sharp}\square^2)^{\square} + (\square^{\sharp}\square^2)^{\sharp\square} \geq_{\square} 0.$$

Hence $\square^{\sharp}\square^2 \in [\square\square\square]_{\square}$ as required.

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