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EFFECTS OF VARYING SUBSTITUTION PARAMETER (ρ) OF THE CES PRODUCTION FUNCTION ON THE ESTIMATION METHODS: BAYESIAN AND FREQUENTIST APPROACHES

IYANIWURA JOSEPH OLUTAYO, ADEPOJU ABOSEDE ADEDAYO AND ADESINA OLUWASEUN AYOBAMI

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ABSTRACT

Intrinsically nonlinear models are models that cannot be made linear irrespective of the linearization method employed. Statisticians are often interested in estimating the parameters of nonlinear models but are faced with great difficulties since some nonlinear models cannot be solved analytically however, researchers have developed a way out of this difficulty using the Gauss-Newton Method via Kmenta approximation. This paper made use of classical and Bayesian approaches to estimate the Constant Elasticity of Substitution (CES) production function. The Metropolis-within Gibbs Algorithm was used to carry out the analysis as shown in the empirical illustrations and the result showed that the Numerical Standard Error (NSE) is minimal while the posterior estimates converged to the region of the true values making the Bayesian approach more preferred.

KEYWORDS: Intrinsically nonlinear model, Gauss-Newton Method, CES production function, Numerical Standard Error, Metropolis-within Gibbs Algorithm.

INTRODUCTION

A macroeconomic production function is a mathematical expression that describes a systematic relationship between inputs and output in an economy. The CES production function has been used extensively in many areas of economics in the past (Solow (1956) and Arrow *et al.* (1961)). This function assumes that the elasticities of substitution between any two inputs are the same, due to its highly undesirability for empirical applications, the multiple-input CES functions gives room for different (constant) elasticities of substitution between different pairs of inputs that have been proposed. The functional form proposed by Uzawa (1962) has constant Allen Uzawa elasticities of substitution and the functional form proposed by McFadden (1963) has constant Hicks-McFadden elasticities of substitution.

The CES production function due to Arrow et al. (1961) reported that estimates are consistent and Kmenta (1967) used the same estimation procedures applicable to the generalized version of the CES function which is restricted to the case of constant returns to scale, and concluded that the estimates are consistent if the input variables are non-stochastic or, if stochastic disturbance is independent in the production function. Nakamura and Nakamura (2008) confirmed Acemoglu and Zilibotti's (2001) work by using a more general functional form for the intermediates' productivities to show how a general CES production with elasticity above unity can arise from an underlying Cobb-Douglas technology. In their specification, two primary input factors are differentiated over a unit interval of intermediate inputs. Each input used just one of the primary factors. Productivity of the intermediate inputs depends on their position in the interval through a specific functional form. CES production function with an elasticity of exactly 2 in the two primary factors maximizes the profit with the choice of primary factor used for each intermediate input. Noda and Kyo (2011) proposed a new approach in analysis of factor augmenting technical change based on a constant elasticity of substitution (CES) production function. Smoothing priors are introduced in Bayesian linear models constructed to examine the technical changes in Taiwan and South Korea at the macroeconomic level and it was revealed that the Bayesian approach can capture the movements of technical change more rigorously than conventional approaches Noda and Kyo (2015).

Bayesian analysis is used by combining prior and likelihood functions to obtain posterior distributions of functions of interest. This makes estimation of parameters straightforward and reliable. However, there are two difficulties that may arise in working with fully specified macroeconomics models which are the exact form of the likelihood function of interest is generally unknown and its approximation known, this consequently makes posterior analysis impossible to obtain analytically due to the issues of not having a closed form. To resolve these difficulties, the Kmenta approximation (Kmenta 1967) was used to compute the likelihood function for the data given a log-linear approximation to the solution of the theoretical model,

Iyaniwura Joseph Olutayo, Department of Statistics, University of Ibadan, Ibadan, Nigeria.
Adepoju Abosede Adedayo, Department of Statistics, University of Ibadan, Ibadan, Nigeria.
Adesina Oluwaseun Ayobami, Department of Statistics, University of Ibadan, Ibadan, Nigeria.

and conduct posterior analysis using the Metropolis-within Gibbs technique which is a part of the Markov Chain Monte Carlo (MCMC) technique developed in recent Bayesian literatures.

Rolando (2012) proposed a Bayesian Markov Chain Monte Carlo estimation of the capital-labour substitution elasticity in developing countries through prior elicitation and concluded that the Bayesian estimator of the Capital-Labour substitution elasticity was theory consistent, and can be used to properly calibrate computable general equilibrium models. Leon-Ledesmal et. al, (2010) used Monte Carlo simulations to capture production function and its first order conditions jointly to identify the elasticity of substitution given biased technical change. Daan Steenkamp (2016) revealed that negative capital-augmenting technical change in several industries weighed on productivity in New Zealand based on Constant Elasticity of Substitution (CES) production functions that permit varying assumptions about factor augmentation and also allows for industry-specific values of the elasticity of substitution between inputs.

Jakub and Jakub(2015) showed that estimates are consistent which implied that the elasticity of substitution between capital and labor has remained relatively stable, at about 0.8–0.9, from 1948 to the 1980s, followed by a period of secular decline in post-war US economy by generalize the normalized Constant Elasticity of Substitution (CES) production function by allowing the elasticity of substitution to vary isoelastically. Jurgen (2014) addressed the relationship between technical change and the elasticity of substitution between factors of production and showed how the elasticity within a CES production setting can change due to technical change. Miguel and Mathan (2011) showed that allowing firms a choice of CES production techniques through the distribution parameter between capital and labor can result in a new class of production functions that are consistent with a balanced growth path even in the presence of capital augmenting technical progress which produces short-run capital-labor complementarity but yields a long-run unit elasticity of substitution. The idea of MCMC simulation is to let the parameters perform a random walk in parameter space according to a Markov chain set up in such a way that its stationary distribution is the posterior distribution.

The aim of this paper is to investigate the sensitivity of the parameters of the multiplicative error based CES production function to varying sample sizes using the Bayesian and frequentist approaches.

The remaining sections are classified as follows; the Methodology is discussed in Section 2 while the Simulation Study and Discussion of Findings are presented in Section 3 and 4 respectively, Conclusion of the work is given in Section 5.

2. METHODOLOGY

Gauss Newton Method
The Gauss Newton model is of the form

$$y = f(X,\beta) + u \qquad (1)$$

y is a $N \times 1$ vector of observation on response variable, $f(X,\beta)$ is the nonlinear form comprising of the X , an $N \times k$ matrix of observation on explanatory variable and β a $K \times 1$ vector of coefficients of the model, and u an $N \times 1$ vector of disturbance term with mean 0 and variance σ^2 .

The matrix form of the model in equation (1) can be expressed as;

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} X_{11} & X_{21} & X_{31} \\ X_{12} & X_{22} & X_{32} \\ \vdots & \vdots & \vdots \\ X_{1N} & X_{2N} & X_{3N} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}$$

$$y = X\beta + u$$

The Gauss Newton method begins by expanding $f(X_i,\beta) = f_i(\beta)$ using Taylor's series up to the first derivative around a set of initial values, $\beta_j^{0\prime} = (\beta_0^0, \beta_1^0, \beta_2^0)$ and representing the required parameters appropriately. Set $\lambda_j = \beta_j - \beta_j^0$, $Y_i^0 = f_i^0 = f(x_i, \lambda_j^0)$, and setting initial values $\lambda_j^0 = \beta_0^0, \beta_1^0, \beta_2^0$.

Using the OLS method, we obtain the estimates by

$$\hat{\lambda} = \left(Z^{0\prime} Z^0\right)^{-1} Z^{0\prime}(D),$$

where; $D = Y - f^0$, Since, $\lambda_j^0 = \beta_j^1 + \beta_j^0$, the revised estimate of β_j is β_j^1 .

Hence, $\beta_j^1 = \hat{\lambda}_j^0 + \beta_j^0$, the process is repeated to obtai~~ned~~ desired estimates as a general rule.

The CES (Non-linear) Production Function Model with a Multiplicative Error term
Given,

$$y = \gamma\left[\delta x_1^{-\rho} + (1-\delta) x_2^{-\rho}\right]^{-\frac{v}{\rho}} e^u \qquad (2)$$

where, y is the response variable, $\gamma, \delta, \rho \ and \ v$ are the nonlinear regression parameters, $x_1 \ and \ x_2$ are the explanatory variables. u is the error component (well behaved, i.e. $N(0,\sigma^2)$). Also, parameter $\gamma \in [0,\infty)$ determines the productivity, $\delta \in [0,1]$ determines the optimal distribution of the inputs, $\rho \in [-1,0) \cup (0,\infty)$ determines the elasticity of substitution, which is $\sigma = \frac{1}{(1+\rho)}$, and $v \in [0,\infty)$ is equal to the elasticity of scale.

The CES function can be written in the form

$$\ln y = \ln f(X,\gamma) + u \qquad (3)$$

where $f(X,\gamma)$ is a N-vector of functions with ith element given by $f(X_i,\gamma)$ and X_i is the ith row of X and γ is a vector of parameters. The equation (3) can also be expressed as

$$\ln y = \ln[\alpha_1 x_1 + \alpha_2 x_2] + u \qquad (4)$$

Since, in the equation (4), the CES function is still nonlinear in parameters and cannot be solved analytically, that is, despite taking the logarithm of both sides, it is impossible to estimate the parameters with the usual linear techniques.

Hence, the CES function is often approximated by the "Kmenta approximation" (Kmenta 1967) which is then estimated by linear estimation techniques.

The Likelihood function

Using the multivariate Normal density function, the likelihood function for the nonlinear regression model can be written as;

$$p(y/\gamma,h)=\frac{h^{\frac{N}{2}}}{(2\pi)^{\frac{N}{2}}}\left\{\exp\left[-\frac{h}{2}\{y-f(X,\gamma)\}'\{y-f(X,\gamma)\}\right]\right\} \quad \text{.....} \quad (5)$$

Where; $h=\frac{1}{\sigma^2}$ is the error precision, and any form $f(X,\gamma)$ is the mean of the distribution.

Estimating the CES function using Kmenta Approximation

The first order Taylor series approximation around $\rho=0$ is;

$$\ln y \approx \ln\gamma + v\delta\ln x_1 + v(1-\delta)\ln x_2 - v\rho\delta(1-\delta)(\ln x_1-\ln x_2)^2 \quad \text{.....} \quad (6)$$

Derivative with respect to "Gamma"

$$\frac{dy}{d\gamma}=x_1^{v\delta}x_2^{v(1-\delta)}\exp\left(-\frac{1}{2}v\rho\delta(1-\delta)(\ln x_1-\ln x_2)^2\right)e^u \quad \text{.....} \quad (7)$$

Derivative with respect to "Delta"

$$\frac{dy}{d\delta}\approx\gamma v\left(\ln x_1-\ln x_2\right)x_1^{v\delta}x_2^{v(1-\delta)}\left(1-\rho\frac{1-2\delta+v\delta(1-\delta)}{2}(\ln x_1-\ln x_2)\right)e^u \quad \text{.....} \quad (8)$$

Derivative with respect to "v"

$$\frac{\partial y}{\partial v}\approx\gamma e^u x_1^{v\delta}x_2^{v(1-\delta)}\left(\delta\ln x_1+(1-\delta)\ln x_2)-\frac{\rho\delta(1-\delta)}{2}(\ln x_1-\ln x_2)^2(1+v(\delta\ln x_1+(1-\delta)\ln x_2))\right) \quad \text{.....} \quad (9)$$

Derivatives with respect to "Rho"

$$f_\rho(0)=-\frac{1}{2}\delta(1-\delta)x_1^{v\delta}x_2^{v(1-\delta)}\left(\ln x_1-\ln x_2\right)^2 \quad \text{.....} \quad (10)$$

Substituting where necessary in the Taylor series approximation in equation (6), we obtain the Kmenta approximation below;

$$\ln y=\alpha_0+\alpha_1\ln x_1+\alpha_2\ln x_2+\frac{1}{2}B_{11}(\ln x_1)^2+\frac{1}{2}B_{22}(\ln x_2)^2+B_{12}\ln x_1\ln x_2 \quad \text{.....} \quad (11)$$

Where; $\gamma=\exp(\alpha_0)$, $v=\alpha_1+\alpha_2$, $\delta=\frac{\alpha_1}{\alpha_1+\alpha_2}$, $\rho=\frac{B_{12}(\alpha_1+\alpha_2)}{\alpha_1\alpha_2}$, and $B_{12}=-B_{11}=-B_{22}$

For the purpose of this study, the equation (11) can be simplified as

$$y^*=\alpha_0+\alpha_1x_1^*+\alpha_2x_2^*+B_{11}x_3^*+B_{22}x_4^*+B_{12}x_5^* \quad \text{.....} \quad (12)$$

Where; $y^*=\ln y$, $x_1^*=\ln x_1$, $x_2^*=\ln x_2$, $x_3^*=\frac{1}{2}(\ln x_1)^2$, $x_4^*=\frac{1}{2}(\ln x_2)^2$ and $x_5^*=\ln x_1\ln x_2$ in order to have a model of the form below which can be solved using matrix techniques.

$$y^*=x_i^*\lambda+e_i^* \quad \text{.....} \quad (13)$$

Therefore, $E[y^*]=x_i^*\lambda=f(X,\gamma)$, the mean of equation (13) where; $\lambda'=[\alpha_0\,\alpha_1\,\alpha_2\,B_{11}\,B_{22}\,B_{12}]$, x_i^* is the design matrix of N x K elements and the error component is well behaved, i.e. $e_i^*\sim N(0,h^{-1}I_N)$ now the method of the OLS can be used directly to obtain the parameters of the equation (11), then substituted to their respective representations at the initial stating of the equation (11) to get the values of the estimates of the model, but of concern in this study is the Bayesian approach.

The following formulae is used to obtain the likelihood

$s^2 = \frac{(y^* - x^*\hat{\lambda})'(y^* - x^*\hat{\lambda})}{v}$, $v = N - K$, N is the sample size and K is the number of parameters.

$\hat{\lambda} = (x^{*\prime} x^*)^{-1} x^* y^*$, where; x^* the design matrix and y^* response variables

The Likelihood Function of the Redefined variables

Since the form of $f(X,\gamma)$ is known to be $f(x^*,\lambda)$ which is linear in parameter and variables, then from equation (12) the likelihood function for this study becomes

$$p(y^*/\lambda,h) = \frac{h^{\frac{N}{2}}}{(2\pi)^{\frac{N}{2}}}\left\{\exp\left[-\frac{h}{2}\left\{y - x^*\hat{\lambda}\right\}'\{y - x^*\hat{\lambda}\}\right]\right\} \qquad \text{.................................} \qquad (14)$$

Where; $\hat{\lambda} = (x^{*\prime} x^*)^{-1} x^* y^*$, x^* the design matrix and y^* response variables and $h = \frac{1}{\sigma^2}$ still the error precision.

The Prior of the Redefined variables

The Independent Normal-Gamma prior is employed for this study depending on the form of $f(X,\gamma)$ which is only investigated using its non-informative aspect.
Therefore, from the law of independent random variables we have that

$$P(\lambda,h) = P(\lambda)\cdot P(h)$$

Where, $P(\lambda) \sim Normal$ and $P(h) \sim Gamma$

$$P(\lambda) = \frac{1}{(2\pi)^{\frac{k}{2}}}|\underline{V}|^{-\frac{1}{2}}\exp\left[-\frac{1}{2}(\lambda - \underline{\lambda})'\underline{V}^{-1}(\lambda - \underline{\lambda})\right]$$

and

$$P(h) = C_G^{-1} h^{\frac{\nu-2}{2}}\exp\left(\frac{-h\underline{\nu}}{2\underline{s}^{-2}}\right)$$

Where, C_G^{-1} is an integrating constant, it is deduced that: $E[\lambda/y^*] = \underline{\lambda}$ is the prior mean of λ and $Var(\lambda/h) = \underline{V}$ is the prior covariance matrix of λ with the mean of h, as $\underline{s}^{-2}$ and $\underline{\nu}$ degree of freedom.

The Posterior of the Redefined variables

Let the Posterior (which is proportional to prior times likelihood) be denoted by $P(\lambda,h/y^*)$.

Mathematically, using $P(\lambda,h/y^*) = P(y^*/\lambda,h)\cdot P(\lambda)\cdot P(h)$,

But note that $P(\lambda,h/y^*) \neq P(\lambda/y^*,h)\cdot P(h/y^*,\lambda)$

Then, the posterior:

$$P(\lambda,h/y^*) = \frac{h^{\frac{N}{2}}}{(2\pi)^{\frac{N}{2}}}\exp\left[-\frac{h}{2}\left(y^* - x^*\hat{\lambda}\right)'\left(y^* - x^*\hat{\lambda}\right)\right]\cdot\frac{1}{(2\pi)^{\frac{k}{2}}}|\underline{V}|^{-\frac{1}{2}}\exp\left[-\frac{1}{2}(\lambda - \underline{\lambda})'\underline{V}^{-1}(\lambda - \underline{\lambda})\right]\cdot C_G^{-1}h^{\frac{\underline{\nu}-2}{2}}\exp\left(\frac{-h\underline{\nu}}{2\underline{s}^{-2}}\right)$$

$$P(\lambda,h/y^*) \propto \exp\left[-\frac{1}{2}\{h\left(y^* - x^*\hat{\lambda}\right)'\left(y^* - x^*\hat{\lambda}\right) + (\lambda - \underline{\lambda})'\underline{V}^{-1}(\lambda - \underline{\lambda})\}\right]\cdot h^{\frac{N+\underline{\nu}-2}{2}}\exp\left(\frac{-h\underline{\nu}}{2\underline{s}^{-2}}\right) \text{..........................} \qquad (15)$$

This joint posterior density for λ and h does not take any well-known distributional form; so it cannot be solved analytically but only through a posterior simulation method.
By ignoring the terms that do not involve λ in equation (15) we obtain,

$$P(\lambda/y^*,h) \propto \exp\left[-\frac{1}{2}\{(\lambda-\bar{\lambda})'\bar{V}^{-1}(\lambda-\bar{\lambda})\}\right] \quad \ldots\ldots \quad (16)$$

Which implies that $\lambda / y,h \sim N(\bar{\lambda},\bar{V})$, a Multivariate Normal density

Where, $\bar{V} = (\underline{V}^{-1} + hx^{*\prime} x^*)^{-1}$ and $\bar{\lambda} = \bar{V}(hx^{*\prime} y^* + \underline{V}^{-1}\underline{\lambda})$

Similarly, by treating equation (15) as a function of h ignoring terms that do not involve h we can obtain

$$P(h/y^*,\lambda) \propto h^{\frac{N+\underline{v}-2}{2}} \exp[-\frac{h}{2}\{(y^*-x^*\lambda)'(y^*-x^*\lambda)+\underline{v}\underline{s}^2\}] \quad \ldots\ldots \quad (17)$$

This also implies that $h/y^*,\lambda \sim G(\bar{s}^{-2},\bar{v})$, a Gamma density

Where, $\bar{v} = N + \underline{v}$ and $\bar{s}^{-2} = \frac{(y^*-x^*\lambda)'(y^*-x^*\lambda)+\underline{v}\underline{s}^2}{\bar{v}}$

The formulae of equations (16) and (17) look familiar to those of the conjugate normal-gamma priors now but it does not relate directly to the posterior of interest. Therefore, the conditional posteriors in equations (16) and (17) do not directly tell us everything about the posterior, $P(\lambda,h/y^*)$. There is a posterior simulator called the Metropolis-Within- Gibbs which makes use of the conditional posteriors to produce random draws $\lambda^{(s)}$ and $h^{(s)}$ for $s = 1,2,\ldots,S$ which can be averaged to produce estimates of the posterior properties just as the Monte Carlo integration.

After obtaining the values of the posterior estimates, then substitute where necessary to obtain the real estimates of the model.

Where; $y^* = \ln y$, $x_1^* = \ln x_1$, $x_2^* = \ln x_2$, $x_3^* = \frac{1}{2}(\ln x_1)^2$, $x_4^* = \frac{1}{2}(\ln x_2)^2$ and $x_5^* = \ln x_1 \ln x_2$

3. Simulation Study

The data used for this paper were generated using a Monte Carlo Simulation technique in which the explanatory variables were drawn from uniform [0,1] distribution independently and the error term obtained from an independent and identical normal distribution with 0 mean and variance 1, the response variable which is the data of interest was obtained from the model by the incorporation of these explanatory variables and the disturbance term.

Table 1: The Multiplicative Error Based CES Production Function for $\rho(0.5)$

Sample Size	True Value	GNM (SE)	KMENTA APPROXIMATION (SE)	POSTERIOR (SD)	NSE
N = 50	$\gamma(1.0)$	3.7924 (2.2863)	0.6997 (0.2295)	1.5867 (0.5683)	0.0090
N=100		10.1134 (2.1808)	0.7754 (0.1564)	0.8927 (0.2718)	0.0043
N=150		2.1722 (0.4637)	0.7559 (0.1297)	0.8130 (0.2337)	0.0037
N=250		1.8019 (0.3121)	1.0616 (0.1264)	0.9729 (0.1585)	0.0025
N=500		1.3197 (0.1175)	0.9123 (0.0780)	0.8509 (0.1049)	0.0017
N=50	$\delta(0.6)$	0.8226 (0.3449)	0.5578 (0.1974)	0.4920 (0.7944)	0.0126
N=100		0.0565 (0.0697)	0.4272 (0.1114)	0.2393 (0.5049)	0.0080
N=150		0.5317 (0.1290)	0.5532 (0.0958)	0.5227 (0.4054)	0.0064
N=250		0.8648 (7.1978)	0.5327 (0.0436)	0.4502 (0.2323)	0.0026
N=500		0.3482 (0.2773)	0.6202 (0.0350)	0.4688 (0.1132)	0.0018
N=50	$\rho(0.5)$	2.6019 (6.2979)	1.6101 (2.0025)	0.9540 (0.3659)	0.0058
N=100		5.3589 (4.9639)	0.4172 (0.7618)	0.0386 (0.1930)	0.0031
N=150		-0.0321 (1.1221)	0.8202 (0.4576)	0.9131 (0.1143)	0.0018
N=250		67.4378 (…)	0.3349 (0.1828)	0.1191 (0.0873)	0.0014
N=500		2.9263 (5.4767)	0.5621 (0.1741)	0.4686 (0.0517)	0.00082
N=50	$v(1.1)$	2.7483 (1.6990)	0.8983 (0.3920)	3.1393 (0.8697)	0.0138
N=100		11.9371 (2.5441)	1.0103 (0.2386)	1.2168 (0.2682)	0.0042
N=150		1.7243 (0.5657)	0.8415 (0.1895)	0.9910 (0.1996)	0.0032
N=250		0.9149 (0.1871)	1.2032 (0.1268)	1.0229 (0.1727)	0.0027
N=500		0.6912 (0.1785)	0.9969 (0.0940)	0.8451 (0.1008)	0.0016

Table 2: The Multiplicative Error Based CES Production Function for $\rho(-0.5)$

Sample Size	True Value	GNM (SE)	KMENTA APPROXIMATION (SE)	POSTERIOR (SD)	NSE
N = 50	$\gamma(1.0)$	3.6905(2.2217)	0.7003(0.2297)	1.5869(0.5683)	0.0090
N=100		9.4318(2.0198)	0.7782(0.1569)	0.8904(0.2720)	0.0043
N=150		1.9430(0.4679)	0.7579(0.1301)	0.8069(0.2335)	0.0037
N=250		1.6830(0.2397)	1.0688(0.1272)	0.9487(0.1587)	0.0025
N=500		1.3128(0.1325)	0.9106(0.0776)	0.8298(0.1043)	0.0016
N=50	$\delta(0.6)$	0.8345(0.3565)	0.5577(0.1976)	0.4920(0.7944)	0.0126
N=100		0.05109(0.0686)	0.4258(0.1113)	0.2400(0.5049)	0.0080
N=150		0.0550(6.2060)	0.5487(0.0986)	0.5557(0.4055)	0.0064
N=250		0.4082(0.1174)	0.5384(0.0441)	0.4360(0.1645)	0.0026
N=500		0.5065(0.0881)	0.6214(0.0362)	0.4702(0.1131)	0.0018
N=50	$\rho(-0.5)$	2.5878(6.4596)	0.4610(1.6025)	0.6296(0.3659)	0.0058
N=100		5.4330(4.8864)	-0.5939(0.6231)	-0.0824(0.1944)	0.0031
N=150		5.3660(2.2180)	-0.2146(0.2873)	-0.0180(0.1144)	0.0018
N=250		0.0045(0.8496)	-0.3997(0.1370)	-0.9129(0.0874)	0.0014
N=500		-0.3736(0.4623)	-0.3410(0.1027)	-0.6633(0.0517)	0.0008
N=50	$v(1.1)$	2.6243(1.6589)	0.8971(0.3921)	3.1360(0.8697)	0.0138
N=100		11.0315(2.3547)	1.0108(0.2385)	1.2039(0.2670)	0.0042
N=150		0.9641(0.2985)	0.8173(0.1895)	0.9608(0.1992)	0.0032
N=250		1.0511(0.2987)	1.1883(0.1267)	0.9420(0.1727)	0.0027
N=500		0.7519(0.1820)	0.9618(0.0937)	0.7640(0.1004)	0.0016

Table 3: The Multiplicative Error Based CES Production Function for $\rho(0)$

Sample Size	True Value	GNM (SE)	KMENTA APPROXIMATION (SE)	POSTERIOR (SD)	NSE
N = 50	$\gamma(1.0)$	2.3234 (0.9431)	0.6999 (0.2295)	1.5885(0.5683)	0.0090
N=100		2.2640 (0.4112)	0.7753 (0.1564)	0.8918(0.2720)	0.0043
N=150		1.6500 (0.3420)	0.7568 (0.1299)	0.8080(0.2337)	0.0040
N=250		1.5223 (0.1994)	1.0640 (0.1268)	0.9592(0.1586)	0.0025
N=500		1.4920 (0.1536)	0.9126 (0.0781)	0.8393(0.1050)	0.0017
N=50	$\delta(0.6)$	0.3411 (0.4378)	0.7655(1.0821)	0.4343(0.7944)	0.0126
N=100		0.0000 (0.0000)	2.4667(5.2400)	-3.3346(0.5049)	0.0080
N=150		4.7890 (4.4470)	0.7210 (0.3371)	1.1085(0.4054)	0.0064
N=250		0.9999 (0.0372)	-0.0744 (1.0146)	1.9114(0.1643)	0.0026
N=500		1.0000(0.0002)	0.2813(0.3332)	0.9724(0.1132)	0.0018
N=50	$\rho(0)$	0.1129 (4.3488)	-6.2847(25.3891)	1.2346(0.3659)	0.0058
N=100		-1.9720(1.1600)	-0.0755(0.4100)	-0.0680(0.0194)	0.003074
N=150		0.0009 (0.0979)	-1.0044 (0.9193)	6.5790(0.1144)	0.0018
N=250		-5.8319(190.5931)	0.7170 (14.1992)	-0.4263(0.0873)	0.0014
N=500		-2.4710(2.5620)	-1.0063(1.1970)	2.1179(0.0244)	0.0004
N=50	$v(1.1)$	0.6248 (0.7100)	-0.2015(0.3921)	2.0419(0.8697)	0.0138
N=100		0.6450 (0.3399)	-0.0904(0.2387)	0.1107(0.2670)	0.0042
N=150		-0.0025 (0.2307)	-0.2644 (0.1895)	-0.1306(0.2000)	0.0032
N=250		-0.0980 (0.1348)	0.0919 (0.1269)	-0.1219(0.1728)	0.0027
N=500		-2.4710(2.5620)	-0.1166(0.0940)	-0.2977(0.1006)	0.0016

Table 4: when ρ is positive $\Rightarrow \sigma < 1$

$\rho = 0.5$	GNM	KMENTA APPROXIMATION	POSTERIOR
N= 50	0.2776	0.3831	0.5118
N=100	0.1573	0.7056	0.9628
N=150	1.0332	0.5494	0.5227
N=250	0.0146	0.7491	0.8936
N=500	0.2547	0.6402	0.6909

Table 5: when ρ is negative $\Rightarrow \sigma > 1$

$\rho = -0.5$	GNM	KMENTA APPROXIMATION	POSTERIOR
N= 50	1.7010	0.6845	0.6136
N=100	0.1554	2.4624	1.0898
N=150	0.1571	1.2732	1.0183
N=250	1.0045	1.6658	11.4811
N=500	1.5964	1.5175	2.9700

Table 6: when ρ is zero $\Rightarrow \sigma = 1$

$\rho = 0$	GNM	KMENTA APPROXIMATION	POSTERIOR
N= 50	0.8986	1.2523	0.4475
N=100	0.6079	1.0994	1.0730
N=150	1.0025	1.3594	0.1319
N=250	0.9020	0.9158	1.7431
N=500	-0.6798	1.1320	0.3207

4. DISCUSSION OF FINDINGS

Table 1 above is a scenario of when $\rho = 0.5$ (positive value) which showed the estimates of parameters of the CES production function with varying sample sizes of 50, 100, 150, 250 and 500 in all cases. The True values of parameters are shown in parenthesis under the column of the true values; the Gauss-Newton Method (GNM) estimates are given with each standard error. It shows that the estimates under the GNM are far from the true values. Therefore, Kmenta approximation has been used to correct the limitations of GNM, a close and careful look at the values of the standard errors under the Kmenta approximation section, showed that the standard errors decreased steadily as sample size increased. Also, Kmenta Approximation produced estimates that are close to the true parameter values while the Posterior estimates seemed to behave better producing estimates that are very close to the true values and the Numerical Standard Error (NSE) decreased steadily as sample size increased.

Table 2 is a scenario of when $\rho = -0.5$ (Negative value) which showed the estimates of parameters of the CES production function with varying sample sizes of 50, 100, 150, 250 and 500 in all cases. The true values of parameters are shown in parenthesis under the column of the true values; the Gauss-Newton Method (GNM) estimates shown as well with each standard error. It shows that the estimates under the GNM are not close the true values. The Kmenta approximation produced estimates that are close to the true values, it is also obvious that the standard errors decreased steadily as sample size increased. The estimates of the parameters of rho (ρ) do not behave well for instance; under Kmenta Approximation and Posterior where N=50, the estimates produced are 0.4610 and 0.6296 respectively which are positive as opposed to the negative true value used or set.

Table 3 above is a scenario of when $\rho = 0$ which showed the estimates of parameters of the CES production function with varying sample sizes of 50, 100, 150, 250 and 500 in all cases. The true values of parameters are shown in parenthesis under the column of the true value; the Gauss-Newton Method (GNM) estimates shown as well with each standard error. It shows that the estimates under the GNM and the Kmenta approximation are not close to the true values as sample size increased. Also, the estimates and Standard error of the parameters of rho (ρ) do not behave well for both the Kmenta Approximation and Posterior results.

From Table 4, When $\rho > 0$ (positive) for any valve of '*v*' (Elasticity of scale parameter) greater than 1 i.e $v > 1$, then with the continual growth in Labour-Capital, the Elasticity of substitution (σ) becomes positively less than 1 for large sample size. The CES production function in classical (Kmenta) and Bayesian approaches do behave well while the GNM does not with the Law of diminishing marginal returns to both Labour and Capital.

From table 5, when $\rho < 0$ (negative) for any value of $'v'$ (Elasticity of scale parameter) greater than 1, i.e. $v > 1$, the Elasticity of substitution (σ) exceed 1, measured the varying factors substituted. The CES production function for GNM, Kmenta and Bayesian approach does not behave well with the Law of diminishing marginal returns to both Labour and Capital. From table 6, when $\rho = 0$, for any value of 'v' (Elasticity of scale parameter) greater than 1($v > 1$) with the constant Elasticity of substitution($\sigma = 1$).The CES production function for GNM and Bayesian approach produced values that are greater than 1 andless than 1.This implies that the elasticity of substitution obtained do not behave well. On the contrary, Kmenta's elasticity of substitution does behave well.

CONCLUSION

In Conclusion, the Bayesian Approach seems to be the most preferred in the sense that the posterior estimates produced are closer to the true values; with consistent decrease in the NSE as sample size increased. It is obvious that when $\rho > 0$ for any value of 'v'(Elasticity of scale parameter) greater than 1, i.e. $v > 1$, the Elasticity of substitution (σ) produced consistent values for both classical and Bayesian approaches while for $\rho < 0$ and $\rho = 0$ for any value of 'v'(Elasticity of scale parameter) greater than 1, then the Elasticity of substitution (σ) produced inconsistent values.

From literatures, the rho (ρ) has always been the nuisance parameter of the CES production function, producing values that are outliers, also past researchers (Iyaniwura, (1974)) who worked on CES (classical approach) had some similar agreement that the determinant of the Elasticity of Substitution, (ρ) tend to pose much problems producing ambiguous values. Similarly, we encountered same challenge when using the classical approach; the reason for pegging (omitting some values of ρ) some values of rho but the Bayesian approach took care of this limitation; which makes the Bayesian Approach more suitable approach.

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