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ON THE PARAMETER ESTIMATION OF FIRST ORDER IMA MODEL CORRUPTED WITH A R (1) ERRORS

D. ENI AND S. A. MAHMUD

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ABSTRACT

In this paper, we showed how autocovariance functions can be used to estimate the variances of the white noises that characterize the IMA (1) models corrupted with AR (1) errors. This was used to develop an iteration formula that can be used to estimate the parameters of the IMA (1) model. We performed simulation studies to demonstrate our findings. The studies showed that our method very closely estimate the true parameters of the process

KEYWORDS: ARMA, IMA, AR, Autocovariance Function, Parameter Estimation

1.0 INTRODUCTION

Consider the first order moving average model

$$w_t = a_t + \theta\, a_{t-1} \quad (1)$$

where,

w_t is an unobserved process of interest

a_t is a white noise process. ie, it is distributed with zero mean and constant variance σ_a^2 and $a_{t-i}, i = 1,2,3..$ are its values at time t-i

θ is a weight parameter (Box and Jenkins1976)

Equation (1) is said to be invertible if the expansion $(1-\theta)^{-1}$ converges in square mean and this is the case where, $|\theta| < 1$, Priestley(1971)

Our interest is the case where equation (1) is not invertible ie $\theta > 1$ but would be through the transformation $w_t - w_{t-1}$. Doing this will result to the first order integrated moving average IMA (1) model of the form (Box and Jenkins1976)

$$(1-L)w_t = a_t + (\theta - 1)a_{t-1} \quad (2)$$

where L is a backward operator ($La_t = a_{t-1}$).

Further more, we postulate the case where w_t can be estimated by an observable process z_t through $w_t = z_t - b_t$, where b_t is an error component introduced by faulty measurement or observation processes and is a an autoregressive process of order one ie AR(1).

Substituting $w_t = z_t - b_t$ into equation (2), we have

$$(1-L)z_t = (1 + (1-\theta)L)a_{t-1} + (1-L)b_t$$

or

$$(1-L)z_t = (1-\phi L)a_{t-1} + (1-L)b_t \quad (3)$$

where

$$\phi = -(1-\theta) \;or\; \theta = 1 + \phi \quad (3b)$$

Since b_t is AR(1),

$$b_t = (\frac{e_t}{(1-\alpha L)} \text{ (Hamilton (1994))}$$

and substituting into equation (3),we have

$$(1-\alpha L)(1-L)z_t = (1-\alpha L)(1-\phi L)a_t + (1-L)e_t$$
$$z_t = (1+\alpha)z_{t-1} - \alpha z_{t-2} + a_t - (\alpha+\phi)a_{t-1} + \alpha\phi a_{t-2} + e_t - e_{t-1} \quad (4)$$

D. Eni, Department of Mathematics/Statistics, Cross River University of Technology, Calabar, Nigeria.
S. A. Mahmud, General Studies Department, Federal Polytechnic, Bauchi, Nigeria.

where

e_t is a white noise process uncorrelated with a_t.

Our interest is to estimate the parameters, θ in w_t and α in b_t through z_t.

The maximum likelihood estimates for the case where both σ_a^2 *and* σ_e^2 are known (the so called "over verification case") are estimated by Barnett (1967) by directly solving the likelihood equation. Chan and Mak (1979) obtained the maximum likelihood estimates for the case where both σ_a^2 *and* σ_e^2 are unknown and where the observations are replicated.

Our interest is to use autocovariance function to estimate the parameter values of the real IMA(1)series as well as the parameter value of the AR(1) errors even where the ratio $\lambda = \frac{\sigma_a^2}{\sigma_e^2}$ is unknown. Eni, et al(2007a) have used the same method to isolate errors of of AR(1) corrupted with MA(1) process. Also Eni, et al (2007b) have considered the case of IMA (1) with white noise. In a similar case, Eni (2006) has considered the case of GARCH (1,1) model with white noise errors using the proposed method.

2.0 VARIANCES OF THE WHITE NOISE PROCESSES

Theorem1

The variances σ_a^2 *and* σ_e^2 of the white noises a_t and e_t respectively are

$$\sigma_a^2 = \frac{\alpha^2 v_0 - \alpha(1+\alpha)v_1 + \alpha v_2}{(1-\alpha)\{\alpha\phi(1-\phi)-(\alpha+\phi)\} + \{1-(\alpha+\phi)(1-\phi)+\alpha\phi(1-\phi)\}}$$

$$\sigma_e^2 = \frac{\{v_0-(1+\alpha)v_1+\alpha v_2\}\{\alpha\phi(1-\phi)-(\alpha+\phi)\}-(1+\alpha)(v_1-v_0)\{1-(\alpha+\phi)(1-\phi)+\alpha\phi(1-\phi)\}}{(1-\alpha)\{\alpha\phi(1-\phi)-(\alpha+\phi)\}+\{1-(\alpha+\phi)(1-\phi)+\alpha\phi(1-\phi)\}}$$

for IMA (1) process corrupted with AR (1) errors

Proof

Multiply through equation (4) by z_t and then take expectations to get the variance v_0 of the process

$$v_0 = (1+\alpha)v_1 - \alpha v_2 + \sigma_a^2 + \sigma_e^2 - (\alpha+\phi)E(z_t a_{t-1}) + \alpha\phi E(z_t a_{t-2}) - E(z_t e_{t-1}) \quad (5)$$

where

$$E(z_t a_t) = \sigma_a^2$$
$$E(z_t e_t) = \sigma_e^2 \quad (6)$$
$$E(z_t z_{t-i}) = v_i$$

See for example Hamilton (1994).

$$E(a_t a_{t-i})\, or\, E(e_t e_{t-i}) = \begin{cases} \sigma_a^2\ or\ \sigma_e^2\ respectively\ for\ i = 0 \\ 0 \qquad\qquad\qquad\qquad\qquad i \neq 0 \end{cases}$$

$$E(a_t e_{t-i}) = 0 \qquad a_t\ and\ e_t\ are\ independent$$

See Box and Jenkins (1976) for example. (7)

Multiply through equation (4) by a_{t-1}, a_{t-2}, e_{t-1} and then take expectations using the set of equations in (6) and (7) to get

$$E(z_t a_{t-1}) = (1-\phi)\sigma_a^2 \quad (8)$$
$$E(z_t a_{t-2}) = (1-\phi)\sigma_a^2 \quad (9)$$
$$E(z_t e_{t-1}) = \alpha\sigma_e^2 \quad (10)$$

Substituting equations (8),(9).and (10) into (5), we obtain

$$v_0 - (1+\alpha)v_1 + \alpha v_2 = \{1-(\alpha+\phi)(1-\phi)+\alpha\phi(1-\phi)\}\sigma_a^2 + (1-\alpha)\sigma_e^2 \quad (11)$$

Multiply through equation (4) by z_{t-1} and then take expectations using the equations in (6), (7), (8), (9) and (10) to get the autocovariance

$$(1+\alpha)(v_1 - v_0) = \{\alpha\phi(1-\phi)-(\alpha+\phi)\}\sigma_a^2 - \sigma_e^2 \quad (12)$$

Solving equations (11) and (12) simultaneously for σ_a^2 *and* σ_e^2, we obtain the required results as

$$\sigma_a^2 = \frac{\alpha^2 v_0 - \alpha(1+\alpha)v_1 + \alpha v_2}{(1-\alpha)\{\alpha\phi(1-\phi)-(\alpha+\phi)\} + \{1-(\alpha+\phi)(1-\phi)+\alpha\phi(1-\phi)\}} \tag{13}$$

$$\sigma_e^2 = \frac{\{v_0-(1+\alpha)v_1+\alpha v_2\}\{\alpha\phi(1-\phi)-(\alpha+\phi)\}-(1+\alpha)(v_1-v_0)\{1-(\alpha+\phi)(1-\phi)+\alpha\phi(1-\phi)\}}{(1-\alpha)\{\alpha\phi(1-\phi)-(\alpha+\phi)\}+\{1-(\alpha+\phi)(1-\phi)+\alpha\phi(1-\phi)\}} \tag{14}$$

In practice, the observed process (4) will be identified as the ARMA (2,2) model

$$z_t = \beta_1 z_{t-1} - \beta_2 z_{t-2} + U_t - \Omega_1 U_{t-1} + \Omega_2 U_{t-2} \tag{15}$$

where

U_t is a white noise process

Comparing equation (4) with (15), we note

$$\beta_1 = 1+\alpha$$
$$\beta_2 = \alpha \tag{15b}$$

$$U_t - \Omega_1 U_{t-1} + \Omega_2 U_{t-2} = (a_t + e_t) - \{ (\alpha+\phi)a_{t-1} + e_{t-1} \} + \alpha\phi a_{t-2} \tag{15c}$$

We group the white noise processes in (15c) according to time t-i, i=0,1,2 to get

$U_t = a_t + e_t$ (i)

$\Omega_1 U_{t-1} = (\alpha+\phi)a_{t-1} + e_{t-1}$ (ii)

$\Omega_2 U_{t-2} = \alpha\phi a_{t-2}$ (iii)

We can estimate $\beta_1, \beta_2,, \Omega_1$ and Ω_2 through the maximum likelihood estimation (Box and Jenkins1976). In this case, the only unknown parameter in equation (4) is ϕ and our objective is to estimate it.

Corollary 1

The variance of the observed process is

$$\sigma_U^2 = \frac{A_1 + A_2\{\alpha\phi(1-\phi)-(\alpha+\phi)\} - A_3\{1-(\alpha+\phi)(1-\phi)+\alpha\phi(1-\phi)\}}{(1-\alpha)\{\alpha\phi(1-\phi)-(\alpha+\phi)\}+\{1-(\alpha+\phi)(1-\phi)+\alpha\phi(1-\phi)\}} \tag{16}$$

where

$$A_1 = \{\alpha^2 v_0 - \alpha(1+\alpha)v_1 + \alpha v_2\}$$
$$A_2 = \{v_0 - (1+\alpha)v_1 + \alpha v_2\}$$
$$A_3 = (1+\alpha)(v_1 - v_0)$$

Proof

Consider from (i)

$$U_t U_t = (a_t + e_t)(a_t + e_t)$$
$$= a_t a_t + 2a_t e_t + e_t e_t$$

Taking expectations, we have

$\sigma_U^2 = \sigma_a^2 + \sigma_e^2$ $\quad a_t$ *and* e_t are independent (17)

Substituting equations (13) and (14) into (17), we have the required result (16)

3.0 Parameter Estimations of IMA (1) Process Corrupted With AR (1) Errors

Theorem 2: The parameter ϕ found in the IMA (1) process corrupted by AR (1) Process can be estimated using the iterative formula below

$$\phi_{i+1} = \phi_i - \left(\frac{P(\phi) - Q(\phi)}{R(\phi) - S(\phi)} \right)\Bigg|_i$$

where

$$P(\phi) = C_2[(1-\alpha)\{\alpha\phi(1-\phi)-(\alpha+\phi)\}+\{1-(\alpha+\phi)(1-\phi)+\alpha\phi(1-\phi)\}]$$
$$Q(\phi) = C_1[A_1 + A_2\{\alpha\phi(1-\phi)-(\alpha+\phi)\}-A_3\{1-(\alpha+\phi)(1-\phi)+\alpha\phi(1-\phi)\}]$$
$$R(\phi) = C_2[(1-\alpha)\{\alpha(1-2\phi)-1\}+\{\alpha(1-2\phi)-(1-\alpha-2\phi)\}]$$
$$S(\phi) = C_1[A_2\{\alpha(1-2\phi)-1\}-A_3\{\alpha(1-2\phi)-(1-\alpha-2\phi)\}]$$
$$A_1 = \{\alpha^2 v_0 - \alpha(1+\alpha)v_1 + \alpha v_2\}$$
$$A_2 = \{v_0 - (1+\alpha)v_1 + \alpha v_2\}$$
$$A_3 = (1+\alpha)(v_1 - v_0)$$
$$C_1 = 1 - \Omega_1(\beta_1 - \Omega_1) + \Omega_2(\beta_1 - \Omega_1)\beta_1 - \Omega_2(\beta_2 - \Omega_2)$$
$$C_2 = v_0 - \beta_1 v_1 + \beta_2 v_2$$

The stating point of the iteration is

$$\phi_0 = \Omega_1 - \beta_1$$

Proof:

Multiply through equation (15) by z_t and then take expectations to get

$$v_0 = \beta_1 v_1 - \beta_2 v_2 + \sigma_U^2 - \Omega_1 E(z_t U_{t-1}) + \Omega_2 E(z_t U_{t-2}) \tag{18}$$

where

$$E(z_t U_t) = \sigma_U^2$$

Multiply through equation (15) by U_{t-1}, U_{t-2} and then take expectations to get

$$E(z_t U_{t-1}) = (\beta_1 - \Omega_1)\sigma_U^2 \tag{19}$$

$$E(z_t U_{t-2}) = \{(\beta_1 - \Omega_1)\beta_1 - (\beta_2 - \Omega_2)\}\sigma_U^2 \tag{20}$$

Substituting equations (19) and (20) into (18), we obtain

$$v_0 - \beta_1 v_1 + \beta_2 v_2 = \{1 - \Omega_1(\beta_1 - \Omega_1) + \Omega_2(\beta_1 - \Omega_1)\beta_1 - \Omega_2(\beta_2 - \Omega_2)\}\sigma_U^2 \tag{21}$$

Substituting equations (16) into (21), we get

$$P(\phi) = Q(\phi)$$

or

$$F(\phi) = P(\phi) - Q(\phi) \tag{22}$$

where

$$P(\phi) = C_2[(1-\alpha)\{\alpha\phi(1-\phi)-(\alpha+\phi)\}+\{1-(\alpha+\phi)(1-\phi)+\alpha\phi(1-\phi)\}]$$
$$Q(\phi) = C_1[A_1 + A_2\{\alpha\phi(1-\phi)-(\alpha+\phi)\}-A_3\{1-(\alpha+\phi)(1-\phi)+\alpha\phi(1-\phi)\}]$$
$$C_1 = 1 - \Omega_1(\beta_1 - \Omega_1) + \Omega_2(\beta_1 - \Omega_1)\beta_1 - \Omega_2(\beta_2 - \Omega_2)$$
$$C_2 = v_0 - \beta_1 v_1 + \beta_2 v_2$$

Our objective is to estimate the parameter ϕ .However; equation (22) is non-linear and can be solved by the Newton-Raphson process. In this case, the ϕ_{i+1} solution may be obtained from the i[th] approximation according to

$$\phi_{i+1} = \phi_i - \left(\frac{P(\phi) - Q(\phi)}{R(\phi) - S(\phi)}\right)\Bigg|_i \tag{23}$$

where

$$P(\phi) = C_2[(1-\alpha)\{\alpha\phi(1-\phi)-(\alpha+\phi)\}+\{1-(\alpha+\phi)(1-\phi)+\alpha\phi(1-\phi)\}]$$
$$Q(\phi) = C_1[A_1 + A_2\{\alpha\phi(1-\phi)-(\alpha+\phi)\}-A_3\{1-(\alpha+\phi)(1-\phi)+\alpha\phi(1-\phi)\}]$$
$$R(\phi) = P(\phi)' = C_2[(1-\alpha)\{\alpha(1-2\phi)-1\}+\{\alpha(1-2\phi)-(1-\alpha-2\phi)\}]$$
$$S(\phi) = Q(\phi)' = C_1[A_2\{\alpha(1-2\phi)-1\}-A_3\{\alpha(1-2\phi)-(1-\alpha-2\phi)\}]$$

$$A_1 = \{\alpha^2 v_0 - \alpha(1+\alpha)v_1 + \alpha v_2\}$$
$$A_2 = \{v_0 - (1+\alpha)v_1 + \alpha v_2\}$$
$$A_3 = (1+\alpha)(v_1 - v_0)$$
$$C_1 = 1 - \Omega_1(\beta_1 - \Omega_1) + \Omega_2(\beta_1 - \Omega_1)\beta_1 - \Omega_2(\beta_2 - \Omega_2)$$
$$C_2 = v_0 - \beta_1 v_1 + \beta_2 v_2$$

We start iteration with $\phi_0 = \Omega_1 - \beta_1$. We obtain this by examining (15b) and (ii) in section 2 We expect the values of ϕ to be such that $f(\phi) = 0$ at the point of convergence.

4.0 ILLUSTRATION

We used the NORMRND facilities in MATLAB5 (1999) to generate the white noise error a_t with mean 0 and variance 2.5. We use this to simulate the values of the non-invertible w_t following equation(1) We do this by using a recursion derived from equation(1) as follows;

$$w_0 = a_0 = 0$$
$$w_1 = a_1$$
$$w_2 = a_2 + \theta w_1 \qquad \Rightarrow a_2 = w_2 - \theta w_1$$
$$w_3 = a_3 + \theta w_2 - \theta^2 w_1 \qquad \Rightarrow a_3 = w_3 - \theta w_2 + \theta^2 w_1$$

Continuing this way, we have

$$w_4 = a_4 + \theta w_3 - \theta^2 w_2 + \theta^3 w_1$$
$$\vdots$$
$$w_t = a_t + \sum_{i=1}^{t-1} (-1)^{i+1} \theta^i w_{t-i} \qquad (24)$$

We chose $\theta = 1.46$ to ensure non- invertibility. (See Priestley 1971)
Next, we generate the IMA form as in equation (2) using the result in the recursion (24) as

$$w_2 - w_1 = (1-L)w_2 = a_2 - a_1 + \theta w_1$$
$$(1-L)w_3 = a_3 - a_2 + \theta w_2 - \theta(1+\theta)w_1$$
$$\vdots$$
$$(1-L)w_t = a_t - a_{t-1} + \theta w_{t-1} - \left(\sum_{i=1}^{t-2} (-1)^{i+1} \theta^i (1+\theta) w_{t-1-i} \right) \qquad (25)$$

Next, we again used the NORMRND facilities in MATLAB5 (1999) to generate the white noise error e_t with mean 0 and variance 1.5. We use this to simulate the values of the error component b_t following equation (3b) We do this by using a recursion derived from equation (3b) as follows.

$$b_1 = e_1$$
$$b_2 = e_2 + \alpha e_1$$
$$\vdots \qquad (26)$$
$$b_t = e_t + \alpha e_{t-1} + \alpha^2 e_{t-2} + \cdots + \alpha^{t-1} e_1$$
$$= \sum_{i=0}^{t-1} \alpha^i e_{t-i}, \quad t = 1,2,...$$

We chose $\alpha = 0.75$ to ensure stationarity. (See Priestley 1971)
We sum the result of recursions (25) and (26) to get the observe set of data, z_t since from equation (3) $w_t = z_t - b_t$.
Our aim is to estimate the parameters, θ in w_t and α in b_t through z_t using theories formulated in sections 2 and 3 of this paper. We compute the first three-autocovariance values of z_t using (Box and Jenkins1976)

$$v_i = \frac{1}{N} \sum_{t=1}^{N-i} (z_t - \mu)(z_{t-i} - \mu).$$

where

$$i = 0,1,2$$

$$\mu = \frac{1}{N}\sum_{t=1}^{N} z_t$$

We obtained the result shown below

$$v_0 = 1.6709$$

$$v_1 = 1.103 \quad (27)$$

$$v_2 = 0.2416$$

We used the McLeod and Sales (1983) maximum likelihood estimates facilities in STATISTICA (1995) to get the following parameter values (found in equation (15));

$$\beta_1 = 1.74$$

$$\beta_2 = 0.736$$

$$\Omega_1 = 2.17 \quad (28)$$

$$\Omega_2 = 0.361$$

The iterative formula (23) was used to estimate the parameter value ϕ of the IMA (1) process with $\phi_0 = 0.43$ as starting value for the iteration. The iteration converges after four attempts to ϕ=0.454. From this, we obtain $\theta = 1 + \phi = 1.454$ (see equation (3)) .This value is very close to the true value of θ=1.46 (see equation (24)) .

Additionally, the value of α found in the AR (1) error process is easily seen to be $\beta_2 = \alpha = 0.736$ (see equation 15b).Again this is close to the true value $\alpha = 0.75$ (see equation (26)).

These results show that the parameters of both the IMA (1) process and the AR (1) errors have been correctly estimated by our method.

5.0 CONCLUSION

We have shown that autocorrelation functions can be used to estimate the true parameters of an IMA (1) process corrupted with AR (1) errors

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