GLOBAL JOURNAL OF PURE AND APPLIED SCIENCES VOL. 13, NO. 4, 2007: 579 - 584



# THE ECONOMIC LOT SIZE FOR RANDOM DEMAND AND QUANTITY IN STORE: A SIMPLE OPTIMAL SOLUTION

F. E. U. OSAGIEDE AND C. O. OGBA

(Received 20 April 2007; Revision Accepted 11 June 2007)

ABSTRACT

An analytical model for determining the optimal inventory replenishment schedule with random demand and quantity in store is proposed. We develop an exact method for computing the logistical system performance measures needed for calculating the quantities, the time interval and the optimal number of replenishment for the system. We show that, the cost function has a minimum turning value. The objective is to minimize the total inventory cost for the system.

**KEYWORDS:** Inventory model, delay cost, random demand, deliveries.

## 1.0 INTRODUCTION

The basic Economic Order Quantity model is a convenient way of determining the optimum order quantity when the demand rate is constant. Donaldson (1977) relaxed the assumption of constant demand with a linear trend in demand. For mathematical connivance, Donaldson analytical solution considered demand bounded by a finite horizon and the model required substantial amount of computational effort to obtain the replenishments. Silver (1979) considered approximated solution procedure for the general optimal times of case of deterministic time varying demand. Ritchie (1984) developed an exact solution by extending the time horizon and simplified the economic lot size formula. Downs *et al.* (2001) consider inventory multiple products with lost sales. Osagiede and Omosigho (2005) developed a computer aided analytical solution for inventory problems with linear increasing demand. Wang and Gerchak (1996); Bollapragada and Morton (1999), and Gurnani *et al.* (2000) considers inventory problem with random yield and random demand. In all these they did not consider the delays between orders and deliveries. Luca (2003) consider the problem in which a facility periodically orders a product from another facility to face a constant demand. The aim was to determine a policy that minimizes the sum of ordering and inventory costs over infinite time horizon. A cursory review can be found in Osagiede (2007). Herbert (2005) presented the optimal inventory policies when sales are discretionary; and analyze the optimal policy to meet a fraction of demand under the classical conditions. Chum and Chue (2005) demonstrated the application of optimal control theory to product pricing and warranty with free replacement under the influence of basic life time distributions. Hill and Pakkala (2005) consider a discounted cash flow method to the base stock inventory model. Their approach concentrated on cash flows and determines the control policy which maximizes the expected net present value of the cash flows associated with a demand.

In this paper, we relax some of the assumptions of EOQ model by assuming that the rate of demand is random with probability distribution $f(x)$ and the replenishment rate is finite. Also over stock losses and penalties for shortages are substantial, while carrying cost are negligible. In this way, we propose an easy to use and efficient general procedure for obtaining optimum stock level and quantity for a given period of time for the system.

The outline of the paper is as follows. In section 2, we give the notation and Assumption use in the model. Section 3 gives detailed problem descriptions. In section 4, the model formulation and the procedure for determining the optimum stock and quantity to be ordered is presented. A numerical evaluation of the technique is presented in section 5. Section 6 gives the remark. Finally, we give the conclusion and further studies in section 7.

## 2.0 ASSUMPTION AND NOTATIONS

This section presents the assumptions and notations used throughout in this paper.

The demand $x$ is random. Overstock losses and penalties for shortages are substantial, but carrying cost is negligible. Lead time is not zero (that is, the replenishment rate is finite). The replenishment orders $q_i$ for the $n-1$ periods that precede period $n$ are known and have been regularly ordered. $c_1$ is the unit loss per unit for overstocks and let $c_2$ be unit loss for understocks. $p_0$ the taken as the initial inventory (start of the first period); $p_i$ is the inventory at the end of period $i$; $x_i$ be the Demand for period $i$; $q_i$ is the replenishment ordered for period $i$; while $f(x)$ be the Probability distribution for demand $x$; $\alpha$ be the cycle time (time interval) for which the inventory problem is considered. $\alpha$ is partitioned into $n$ equal periods of length of $T_i$.

## 3.0 PROBLEM DESCRIPTION

We consider the inventory problem in a time interval $\alpha$ partitioned into $n$ equal periods $T$. We assume that the lead time is equal to $\alpha$. At the beginning of every period $T$, orders are placed so that deliveries will be made $nth$ periods. Let $f(x)$ be the probability distribution for the demand level $x$, covering a period (interval) $T$. Under this interval $T$, if $x$ is less than inventory $p$, the left-over pieces will be sold at a unit loss of $c_1$; if $x$ is greater the $p$, a special order for the backlog piece is made and the extra cost will be represented by a unit loss of $c_2$. If the carrying cost is not substantial in comparison with $c_1$ *and* $c_2$, the interval $T$ no longer has an effect, and we can say the management policy is time independent.

Let $p$ be the quantity to be placed in store (stock). Two mutually exclusive situations are possible:

(a) $x \le p$: the stock covers the demand and the quantity $p-x$ is sold at a unit loss of $c_1$

F. E. U. Osagiede, Department of Mathematics, University of Benin. Benin City, Nigeria.
C. O. Ogba, Department of Mathematics, University of Benin. Benin City, Nigeria.

(b) $x > p$: shortage exists and $x - p$ pieces must be specially ordered, lading to a unit loss of $c_2$ per unit.

If $f(x)$, the probability distribution for demand $x$ is known, the problem now is find the quantity $q_n$ to be ordered at the $nth$ period.

We now attempt to find the optimum quantities $q_n$ for the system.

## 4.0 MODEL DEVELOPMENT AND SOLUTION

Based on the conditions and assumptions earlier made in this paper, then the formulation is as follows:

$$\left.\begin{aligned} p_1 &= p_0 - x_1 + q_1 \\ p_2 &= p_1 - x_2 + q_2 \\ &\cdots\cdots\cdots \\ p_{n-1} &= p_{n-2} - x_{n-1} + q_{n-1} \\ p_n &= p_{n-1} - x_n + q_n \end{aligned}\right\} \quad (1)$$

But

$$p_n = p_{n-2} - x_{n-1} - x_n + q_{n-1} + q_n \quad (2)$$

and a continuous substitution for $p_i$, $i = n-2, n-3, \cdots, 1$ yield

$$p_n = p_0 + q_1 + q_2 + \cdots + q_n - (x_1 + x_2 + \cdots + x_n) \quad (3)$$

If we set,

$$p = p_0 + q_1 + q_2 + \cdots + q_n \quad (4)$$

$$x = x_1 + x_2 + \cdots + x_n \quad (5)$$

Then equation (3) reduces to

$$p_n = p - x \quad (6)$$

The quantity $p_n$ can either be the following

1. negative if $p < x$ or
2. positive if $p > x$

The mathematical expectation of the total cost for time interval $\alpha = nT$ is given by (see Kaufman 1963)

$$W(p) = c_1 \sum_{x=0}^{p} (p-x) f(x) + c_2 \sum_{x=p+1}^{u} (x-p) f(x) \quad (7)$$

where $u$ represents the highest possible value which can attain.

Note that if we are able to find the value of $p$ that minimizes the cost function (7), then the optimum replenishment $q_n$, can readily be derived from equation (4), after substituting the optimal value of $p$, which is $p^*$ into equation (4).

The common method (the numerical calculation method) used in obtaining $p^*$ from (4) is by computing the values of $W(p)$, using equation (7)for all possible values of $p$. Thereafter, the value $p$ which yields the minimum value for $W(p)$ is chosen as the optimum value of $p^*$. This method has been identified to involve mathematical complexity and time consuming. However, it is the intention of this paper to make known a model that can use to find $p^*$, which avoids the stressful numerical calculations of the method described above.

**Proposition:**

The minimum of the cost function given by equation (7) for $W(p,)$ occurs for a value $p^*$ such that

$$f(x \le p^* - 1) < \lambda < f(x \le p^*)$$

where

$$\lambda = \frac{c_2}{c_1 + c_2} = \text{Shortage or scarcity rate,}$$

and $f(x \le p^*) = f(0) + f(1) + \cdots + f(p^*)$.

**Proof**

Given

$$W(p) = c_1 \sum_{x=0}^{p} (p-x) f(x) + c_2 \sum_{x=p+1}^{u} (x-p) f(x)$$

If we use $p$+1, instead of, the equation above becomes

$$W(p+1) = c_1 \sum_{x=0}^{p+1} (p+1-x) f(x) + c_2 \sum_{x=p+2}^{\infty} (x-p-1) f(x) \quad (8)$$

Expanding equation (8) we have

$$W(p+1) = c_1 \sum_{x=0}^{p} (p+1-x) f(x) + c_2 \sum_{x=p+1}^{u} (x-p-1) f(x) \quad (9)$$

But

$$\sum_{x=p+1}^{u} f(x) = \sum_{x=0}^{u} f(x) - \sum_{x=0}^{p} f(x)$$

$$\sum_{x=p+1}^{u} f(x) = 1 - \sum_{x=0}^{p} f(x) \quad (10)$$

Thus,

$$W(p+1) = c_1 \sum_{x=0}^{p} (p-x) f(x) + c_2 \sum_{x=p+1}^{u} (x-p) f(x) + c_1 \sum_{x=0}^{p} f(x) - c_2 + c_2 \sum_{x=0}^{p} f(x) \quad (11)$$

Substitute $W(p) = c_1 \sum_{x=0}^{p} (p-x) f(x) + c_2 \sum_{x=p+1}^{u} (x-p) f(x)$

into (11) with the expression for $W(p)$ and factorizing terms that have $\sum_{x=0}^{p} f(x)$ in common, we have

$$W(p+1) = W(p) + (c_1 + c_2)\sum_{x=0}^{p} f(x) - c_2$$

But $$\sum_{x=0}^{p} f(x) = f(x \le p)$$

Hence,

$$W(p+1) = W(p) + (c_1 + c_2) f(x \le p) - c_2 \tag{12}$$

In a similar manner;

$$W(p-1) = c_1 \sum_{x=0}^{p-1} (p-1-x) f(x) + c_2 \sum_{x=p}^{u} (x-p+1) f(x)$$

$$= c_1 \sum_{x=0}^{p} (p-1-x) f(x) - c_1 (p-1-p) f(p)$$

$$+ c_2 \sum_{x=p+1}^{u} [x-(p+1)] f(x) + c_2 (p-(p+1)) f(p)$$

Recalling equation (10)

$$W(p-1) = c_1 \sum_{x=0}^{p} (p-x) f(x) + c_2 \sum_{x=p}^{u} (x-p) f(x) - c_1 \sum_{x=0}^{p} f(x)$$

$$+ c_2 \left[1 - \sum_{x=0}^{p} f(x)\right] + c_1 f(p) + c_2 f(p)$$

$$= \left[c_1 \sum_{x=0}^{p} (p-x) f(x) + c_2 \sum_{x=p+1}^{u} (x-p) f(x)\right] - c_1 \left[\sum_{x=0}^{p} f(x) - f(p)\right] + c_2 \left[\sum_{x=0}^{p} f(x) - f(p)\right] + c_2 \tag{13}$$

comparing (13) with the expression for $W(p)$ and taking note that

$$\sum_{x=0}^{p} f(x) - f(p) = f(x \le p-1) \quad \text{, then}$$

$$W(p-1) = W(p) - (c_1 + c_2) f(x \le p-1) + c_2 \tag{14}$$

From our knowledge of the convex nature for the curve of inventory cost function, it follows that, suppose $p^*$ is the optimal stock, then $p^*$ is such that (Osagiede 2002)

$$[W(p^*-1) - W(p^*)][W(p^*) - W(p^*+1)] < 0 \tag{15}$$

Further, it can be more simplified as

$$W(p^*-1) > W(p^*) < W(p^*+1) \tag{16}$$

Which is equivalent to

$$\left.\begin{array}{l} \text{(a)} \quad W(p^*+1) - W(p^*) > 0 \\ \text{(b)} \quad W(p^*-1) - W(p^*) > 0 \end{array}\right\} \tag{17}$$

Equation (17a) implies that equation (12) is

$$(c_1 + c_2) f(x \le p^*) - c_2 > 0 \tag{18}$$

and equation (17b) implies that equation (14) is

$$-(c_1 + c_2) f(x \le p^* - 1) + c_2 > 0 \tag{19}$$

Combining equations (18) and (19), we have,

$$f(x \le p^* - 1) < \frac{c_2}{c_1 + c_2} < f(x \le p^*) \tag{20}$$

If we set $\lambda = \dfrac{c_2}{c_1 + c_2}$, then

$$f(x \le p^* - 1) < \lambda < f(x \le p^*) \tag{21}$$

The value $p^*$ which satisfies (20) gives the value of the stock that minimize $W(p)$.

This equation (21) is now our propose model for an enumeration process for determining $p^*$. The optimal stock $p^*$ having been determined and the quantities $q_1, q_2, \cdots, q_{n-1}$ been known, we have,

$$q_n^* = q_n = p^* - p_0 - (q_1 + q_2 + \cdots + q_{n-1}) \quad (22)$$

(22) was formed from (4) after replacing $p$ by $p^*$. Thus, the optimal replenishment quantity $q_n^*$ can be obtained using (22).

Note that, if the probability distribution remains unchanged, when the time interval $\alpha$ is extended, the optimal quantity of stock $p^*$ will be invariant. Also since the quantity $q_n^*$ has been determined, we can calculate the quantity $q_{n+1}^*$ as

$$\left.\begin{aligned} q_{n+1}^* &= p^* - p_1 - (q_2 + q_3 + \cdots + q_n^*) \\ and& \\ q_{n+2}^* &= p^* - p_2 - (q_3 + q_4 + \cdots + q_{n+1}^*) \\ &\cdots\cdots\cdots \\ q_{n+l}^* &= p^* - p_l - (q_{l+1} + q_{l+2} + \cdots + q_{n+l-1}^*) \end{aligned}\right\} \quad (23)$$

This makes it possible to order the optimum quantities in sufficient time.

This completes our proposed model for determining the optimum replenishment quantity $q_n^*$, when the demand is random. The method is iterative, exact and can be implemented on a computer.

## 5.0 NUMERICAL ILLUSTRATIONS

In this section, we shall illustrate the use of our proposed model with our some hypothetical example. The hypothetical example shall be solved, first, using the numerical calculation method and secondly using our proposed model. The ease associated with the model will be obviously noticed.

**Example:**

A particular warehouse receives stock replenishment supply of a product every month. The product received usually comes in consignments of a particular size. During the first four months of a particular year, replenishments were as given in the table below. The lead time is five months.

**Table 1:** Parameters for the problem.

| | |
|---|---|
| $q_1$ | 1 consignment |
| $q_2$ | ½ consignment |
| $q_3$ | 2 consignment |
| $q_4$ | ½ consignment |
| $p_0$ | 1 consignment |
| $c_1$ | # 80,000 |
| $c_2$ | # 120, 000 |
| $T$ | One month |
| $\alpha$ | 5 months |

The probability distribution of demand $x$, calculated for 5 months, together with the value of $p$, the quantity in store and $x$ are shown on the table below

**Table 2:** Probability distribution for demand $x$

| $p$ | $x$ | $f(x)$ |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 2 | 0.1 |
| 2 | 1 | 0.05 |
| 3 | 4 | 0.15 |
| 4 | 3 | 0.01 |
| 5 | 7 | 0.20 |
| 6 | 6 | 0.25 |
| 7 | 5 | 0.24 |
| 8 or more | 8 | 0 |

The problem is: Find the optimal value $p_5^*$ that should be ordered for the fifth month.

The solution shall be found using two methods:

1. **Method I: Solution by numerical calculation**
2. **Method II: Solution by the proposed model**

$f(x)$ has been carefully selected so as to satisfy the properties of a probability mass function (pmf) and to suit our propose model.

**Method I:** We first of all determine the Economic stock level $p^*$ for the period of five months as follows:

Recall the expression for the total cost function:

$$W(p) = c_1 \sum_{x=0}^{p} (p-x) f(x) + c_2 \sum_{x=p+1}^{u} (x-p) f(x)$$

$c_1$ = # 80,000; $c_2$ = # 120, 000. We now calculate $W(p)$ for $p$ = 0, 1, 2, 3, ...

$$W(0) = 80\left[\sum_{x=0}^{0} (-x) f(x)\right] + 120\left[\sum_{x=1}^{u} x f(x)\right]$$

$$= 80[0(0)] + 120[1(0.05) + 2(0.10) + 3(0.01) + 4(0.15) + 5(0.24) + 6(0.25) + 7(0.2) + 8(0.00)]$$

= # 597.6 Thousand

$$W(1) = 80\left[\sum_{x=0}^{1}(1-x)f(x)\right] + 120\left[\sum_{x=2}^{8}(x-1)f(x)\right]$$

$$= 80[1(0) + 0(0.05)] + 120[1(0.01) + 2(0.01) + 3(0.05) + 4(0.24) + 5(0.25) + 6(0.20) + 7(0.002)]$$

= # 5477.6 Thousand.

Continuing in this manner we have;

$W(2)$ = # = 367.6 Thousand; $W(3)$ = # 277.6 Thousand; $W(4)$ = # 189.6 Thousand; $W(5)$ = # 131.6 Thousand; $W(6)$ = # 121.6 Thousand; $W(7)$ = # 161.6 Thousand and $W(p) \geq 8$ is # 161.6 Thousand.

Since the value of $p$ = 6 gives the minimum value for $W(p)$ then $p^*$ = 6. From this, we find the optimum replenishment quantity $q_5^*$ using (22)

$$q_n^* = q_n = p^* - p_0 - (q_1 + q_2 + \cdots + q_{n-1})$$

$$q_5^* = 6 - 1 - \left(1 + \frac{1}{2} + 2 + \frac{1}{2}\right) = 1$$

Therefore, $q_5^*$ = 1 consignment, this means that at the fifth month one (01) consignment of the product should be ordered.

**Method II:** We first determine the Economic stock level as follows:

Recall from our model that the Economic stock level $p^*$ satisfies the inequality

$$f(x \le p^* - 1) < \lambda < f(x \le p^*)$$

where $\lambda = \frac{c_2}{c_1 + c_2}$.

From our example, $\lambda = \frac{120{,}000}{80{,}000 + 1200{,}000} = 0.6$

We now construct the table for our values obtained.

| $p$ | $x$ | $f(x)$ | $f(x \le p)$ |
|---|---|---|---|
| 0 | 0 | 0 | 0.00 |
| 1 | 2 | 0.1 | 0.05 |
| 2 | 1 | 0.05 | 0.15 |
| 3 | 4 | 0.15 | 0.16 |
| 4 | 3 | 0.01 | 0.31 |
| 5 | 7 | 0.20 | 0.55 |
| 6 | 6 | 0.25 | 0.80 |
| 7 | 5 | 0.24 | 1.00 |
| 8 or more | 8 | 0 | 1.00 |

Thus, the optimal stock level for a period of $\alpha$ (5 months) is obtained as follows:

$$f(x < 5) < \lambda = 0.60 < f(x \le 6)$$

As we did in method I,

$$q_5^* = 6 - 1 - \left(1 + \frac{1}{2} + 2 + \frac{1}{2}\right) = 1$$

$q_5^*$ = 1 consignment (this correspond to the same value as in method I)

So the optimal replenishment quantity $q_5^*$ to be ordered for the fifth month is one consignment.

## 6.0 REMARK:

The use of our model saves time and reduces the computational complexity of calculating the optimal economic stock level $p^*$. In our model, the numerical calculations of the cost function $W(p)$ for possible values of $p$ until you find $p^*$ which minimizes $W(p)$ is needless. Instead apply the model suggested by our proposition. This reduces the computational complexity.

The model is formulated based on the assumption that carrying cost is negligible. The assumption of finite replenishment rate and random demand is deemed to be a better representation of what obtains in the real life situation.

## 7.0 CONCLUSION

The optimal replenishment schedule with random demand and quantity in store is examined. The computational complexity for solving this problem is simplified. A new expression for the cost function and the quantities was derived. The convex property of the total cost function has been used in developing our proposed model. Clearly the new solution procedure reported in this paper is computationally efficient. Work is currently going on using the model but incorporating deteriorating items.

## ACKNOWLEDGEMENT

The authors would like to thank the anonymous referees for their helpful comments and suggestions, which significantly improved the paper and gave more clarity.

## REFERENCES

Bollapragada, S. and Morton, T. E., 1999. Myopic Heuristics for the Random Yield Problem Opl. Res. 47: 713-722.

Donaldson, W. A., 1977. Inventory replenishment policy for a linear trend in demand-An analytical solution. Operational Research Quarterly, 28: 663 - 670.

Downs, B., Metters, R. and Semple, J., 2001. Managing Inventory with Multiple Products, Lags in Delivery, Resource Constraints and Lost Sales. A Mathematical Programming Approach Management Science 47 (3): 464- 479.

Gurnani, H., Akella, R. and Lehoczky, J., 2000. Supply Management in Assembly Systems with Random Yield and Random Demand" I.I.E. Trans 32: 701-714.

Herbert, E. Scarf ., 2005. Optimal Inventory Policies when Sales are Discretionary" Int. Journal of Prod. Economics 93-94, 111-119.

Hill, R. M. and Pakkala, T. P. M., 2005. A Discounted Cash Flow approach to the Base stock Inventory Model Int. Journal of Prod. Economics 93-94, 439-445.

Kaufmann Arnold., 1963. Methods and Models of Operations Research. Prentice-Hall, Inc., Englewood Cliffs, N.J

Luca, B., 2003. Rounding off the Optimal Solution of the Economic Lot Size Problem" Int. Journal of Prod. Economics 81-82, 385-392.

Osagiede F. E. U., 2002. Inventory Control Models for Items with time Dependent Increasing Demand" Ph.D Thesis, University of Benin, Benin City, Nigeria.

Osagiede, F. E. U. and Omosigho S. E., 2005. A Computer Aided Analytical Solution for Inventory Problems with Linear Increasing Demand Int. Journal Inst. Maths and Comp. Sciences (Comp. Sc. Ser.) 16(2): 159-167.

Osagiede, F. E. U., 2007. A Single Stage production system with uncertain demand incorporating shortages. Accepted in the Global Journal of Pure and Applied Sciences.

Pei-Chum, Lin. and Li-Yen, Shue., 2005. Application of Optimal control theory to Product Pricing and Warranty with free Replacement Under the influence of Basic Lifetime Distributions "Int. Journal. Computer and Industrial Engineering 48: 69-82.

Ritchie, E., 1984. The EOQ for Linear Increasing Demand" A Simple Optimal Solution" Journal of Operational Research Soc. 35: 947-952.

Silver, E. A., 1979. A Simple Inventory Replenishment Decision Rule for a Linear Trend in Demand" Journal of Operational Research Soc. 30: 71-75.

Wang, Y. and Gerchak, Y., 1996. Periodic Review Production Models with Variable Capacity, Random Yield and Uncertain Demand" Management Science 42:130-137.