GLOBAL JOURNAL OF PURE AND APPLIED SCIENCES VOL. 10, NO. 3, JUNE 2004: 465 - 471
COPYRIGHT(C) BACHUDO SCIENCE CO. LTD. PRINTED IN NIGERIA. ISSN 1118-0579

A NEW SUMMABILITY PROPERTY OF THE ZEROS OF BESSEL FUNCTIONS

E. C. OBI

(Received 10 April, 2002; Revision Accepted 15 March, 2004)

ABSTRACT

While transformations of the Bessel function $J_v(z)$ and its zeros j_{vk} are known which convert them into semicontinuous matrices that represent regular summability methods stronger than the (C,1)–mean, there is no known transformation which can convert this cylinder function or its roots into a method of summability more efficient than (C,r) when r>1. A solution to this problem for j_{vk} has been found. It is proved that σ_n - summability, is a more efficient method of summability than the classical method of the (C,r) – mean, for all $r \geq 1$ and $n > \frac{r+3}{2}$.

KEY WORDS: Summability, (C,r) –mean, Bessel functions.

1. INTRODUCTION

Let $J_v(z)$ be the Bessel function of the first kind, j_{vk} its zeros ordered by the inequality $|\mathrm{Re}\, j_{vk}| \leq |\mathrm{Re}\, j_{v,k+1}|$, $k \geq 1$, and let (C,r) be the Cesaro mean of integral ordered $r \geq 1$. For r = 1 and some $v \geq 0$, the Cooke-transformation,

$2J^2_{v+k}(\lambda) \equiv a_{\lambda k}$ of $J_v(\lambda)$ represents a conversion of $J_v(\lambda)$ into a regular infinite semicontinuous matrix method of summability consistent with (C,1) [Cooke (1937)]. The (O,m)-mean , $t^{(m)}_{vk} = \beta(m,v)\, j_{vk}^{-2m}$, for a certain rational function $\beta(m,v)$ of v, offers a transformation of j_{vk} into a regular semicontinuous matrix method, consistent also with the (C,1)- mean [cf. Obi (1986)]. It is natural to ask whether there is an analogue of this phenomenon when r>1? Precisely, is there (or how can one construct) a transformation of either the cylinder function or its zeros that will represent a semicontinuous matrix summability method which is more efficient than the classical discontinuous (C,r), when r>1?

In §2, we give a sequence of solutions $g^{[n]}_{vk} \equiv \sigma_n(vk)/\sigma_n(o)$ to the problem concerning the zeros, by manipulating the special function,

$$(1) \qquad \sigma_n(v) = \sum_{k=1}^{\infty} j_{vs}^{-2n} \; (v \geq 0; n \geq 1).$$

This symmetric function of j_{vk} is called the *Rayleigh function* of order $n \geq 1$. Our results are summarized in the theorem of §3. Meanwhile, we first summarize the basic properties of $\sigma_n(v)$.

E. C. OBI, Department of Mathematics, University of Nigeria, Nsuka, Nigeria

The $\sigma_n(v)$ is of interest in many ways. It is rational in v[Kishore(1967)]: $\sigma_n = \phi_n/\pi_n$, where $\pi_n(v) = 4^n \prod_{k=1}^{n}(v+k)^{[n/k]}$, [n/k] being the integral part of n/k, and $\phi_n(v)$ is a polynomial (called the Rayleigh polynomial) whose explicit formula is unknown. The degree $d_n = \deg \phi_n$ of $\phi_n(v)$ has the surprising property that $d_n - d_{n-1}$ is the total number of nontrivial divisors of n, for every n>1. The leading coefficient e_n turns out to be the nth Catalan number (a gem of combinatorial mathematicians). Other curious properties of (1) (which can be found in Carlitz (1963) and Obi (1978)) include its "local" relationships with the Bernoulli and Genocchi numbers at $v = \pm\frac{1}{2}$, and the uncanny congruence properties of the positive integers $a_n = 2^{2n}\, n!(n-1)!\sigma_n(0)$ which caught the attention of Carlitz (1963). The involvement of $\sigma_n(v)$ in the solution of some Riccatti type of differential equations can be found in Kishore (1967). Its role in analytic continuation of holomorphic functions into partial star domains is given in Obi (1987). The real-analytic status of $\sigma_n(v)$ on v>-1 has been given in Obi (1980). In particular it is of the completely monotonic (cm) subclass of real-analytic functions on v>-1, that is, $(-1)^m \sigma_n^{(m)}(v) > 0$ on $v > -1$, $\forall m \geq 0$. The inequality, $\left|\sigma_n^{(m)}(v)\right| \leq sn^m \sigma_n(v)$ (cf. Obi (1975) on $v \geq 0$, where $s \equiv s(n, m) = n(n + 1) \ldots (n+m-1)$, with $s(n, 0) = 1$, will be used in the sequel. In §2 we give the basic lemmas that will be needed for the main theorem.

2. BASIC LEMMAS

Let A and B be two matrix methods of summability. The method A is said to be *at least as efficient* as B if A is consistent with B, and a series W is A-bounded (i.e. its A-transform A(W,x) is a bounded function on x>0) whenever W is B-bounded (cf. Cooke (1950) and Obi (1988)). If A is at least as efficient as B and sums a series not B-summable, we will describe A as being more efficient than B. Now for the Rayleigh function $\sigma_n(u)$, let $g_{uk}^{[n]}$ and $\sigma_n(W,u)$ be the matrix and the series given by

$$(2)\qquad g_{uk}^{[n]} = \frac{\sigma_n(uk)}{\sigma_n(o)}\ ((u,k) \in IR^+ \times Z^+) \text{ and } \sigma_n(W,u) = \sum_{k=1} g_{uk}^{[n]} z_k\ (u>0),$$

where $W \equiv \sum_{k=1}^{\infty} z_k$. If $\lim_{u\to 0} \sigma_n(W,u) \equiv \gamma$ exists, we will call γ the σ_n-sum of W (or declare W as being σ_n-summable to γ).

We will prove in Theorem 1 that σ_n-summability is a more efficient method of summability than the classical method of the (C,r)-mean, $\forall r \geq 1$. The Lemmas of this section are to that end.

Lemma 1 For all n >1,

(3) $0 < g_{uk}^{[n]} < n^{2n-1} / (uk)^{2n-1}$ (u, k>0).

Proof Let us consider the structure theorem of Kishore for the Rayleigh functions (cf. Kishore (1964) whereby σ_n can be expressed in the form,

(4) $$\sigma_n(u) = \sum_{i=1}^{c(n)} \{2^{2n-n_i} p_i(u)\}^{-1}$$

where $n_i \in Z^+$, and c(n) and n_i are such that $\sum_i 2^{n_i} = \frac{1}{n}\binom{2n-2}{n-1} \equiv e_n$ is the n^{th} Catalan number, and

(4a) $$p_i(u) = \prod_{j=1}^{n} (u + b_{ij})^{a_{ij}}, (1 \le b_{ij} \le n; 1 \le a_{ij} \le n)$$

is a polynomial of degree 2n-1 for each i. The b_{ij}, like the a_{ij}, are integers. Now since $\sum_{j=1}^{n} a_{ij} = 2n-1$, then from (4) and (4a), we get, on u >-1,

(4b) $$0 < \sigma_n(u) \le \sum_{i=1}^{c} 2^{n_i - 2n} \left\{ \prod_{j=1}^{n} (u+1)^{-a_{ij}} \right\}$$

$$= 4^{-n} \sum_{i=1}^{c} 2^{n_i} (u+1)^{1-2n} = e_n / \{4^n (u+1)^{2n-1}\}$$

But since $1 \le b_{ij} \le n$, we similarly see from (4a) and (4) that

(4c). $$\sigma_n(u) \ge e_n / \{4^n (u+n)^{2n-1}\} \quad (u>1)$$

Combining (4b-c), we get

(5) $$\frac{e_n}{4^n (u+n)^{2n-1}} \le \sigma_n(u) \le \frac{e_n}{4^n (u+1)^{2n-1}} \quad (u>-1)$$

In (5), consider the first inequality, with u=0; the second with u there replaced by uk (for any u, k>0). It follows that

$$0 < \frac{\sigma_n(uk)}{\sigma_n(0)} \le \frac{n^{2n-1}}{(uk+1)^{2n-1}} \le \frac{n^{2n-1}}{(uk)^{2n-1}}, \text{ as required.}$$

For the next two lemmas, $h_n(u)$ is the first column matrix, $g_{u1}^{[n]}$.

Lemma 2 Let r $\in$ N, and n > $\frac{r+2}{2}$. Then on u > 1, $h_n^{(r+1)}(u)$ is of the order of the function $u^{-(r+a+1)}$, where a is any constant chosen from 0 < a < 2n - (r + 2).

Proof Fix $r \in N$ and $n > \frac{r+2}{2}$. Let $u > 1$, and let

$$s(n, m) = n(n+1)(n+2)\ldots(n+m-1) \text{ if } m \geq 1,\ s(n, 0) = 1.$$

Then (3a) together with Lemma 1 yields

$$\left|h_n^{(r+1)}(u)\right| = \frac{|\sigma_n^{(r+1)}(u)|}{\sigma_n(o)} \leq \frac{s(n,r+1)n^{r+1}\,\sigma_n(u)}{\sigma_n(o)} \tag{6}$$

$$= s(n, r+1)n^{r+1}\, h_n(u) < \frac{s(n,r+1)n^{r+1}\, n^{2n-1}}{u^{2n-1}},$$

the last inequality being due to Lemma 1. Since $2n-(r+2) > 0$ by hypothesis, we can select a number, a, such that $0 < a < 2n-(r+2)$. Multiplying both ends of (6) by u^{r+a+1} we get

$$\text{(6a)}\qquad u^{r+a+1}\left|h_n^{(r+1)}(u)\right| < \frac{s(n,r+1)n^{2n+r}}{u^{2n-(r+2+a)}}.$$

By the choice of a, $2n-(r+2+a) > 0$, and so in the interval $u > 1$ we also have $u^{2n-(r+2+a)} > 1$. Hence on $u > 1$, the right side of (6a) is $< s(n, r+1)n^{2n+r} \equiv b$, a constant, with respect to u. Thus

$$h_n^{(r+1)}(u) = O\left(\frac{1}{u^{r+a+1}}\right), \text{ as required.}$$

Lemma 3 Given $r \in N$ and $n > \frac{r+1}{2}$ we have $h_n(u) = o\left(\frac{1}{u^r}\right)$ as $u \to \infty$.

Proof Lemma 1 applies again: When $u>0$,

$$0 < u^r h_n(u) = u^r g_n^{(n)} < u^r n^{2n-1}/u^{2n-1} = u^{2n-1}/u^{2n-(r+1)} \to 0 \text{ as } u \to \infty,$$ since $2n-(r+1) > 0$ by data.

As usual, $\omega = \sum_{k=0}^{\infty} \omega_k$ will be a given series, $C_m^{(r)}(\omega)$ is its (C,r)-transform, so that $C_m^{(r)} \to$ the (C,r)-sum of ω as $m \to \infty$.

Lemma 4 Given $r \in N$ suppose $h: (0,\infty) \to R$ satisfies

(a) $h \in C^{(r+1)}(0,\infty)$;

(b) for some $a>0$, $h^{(r+1)}(x) = O\left(\frac{1}{x^{r+a+1}}\right)$, on $x > 1$;

(c) $h(x) \to 1$ as $x \to 0$, but $x^r h(x) \to 0$ as $x \to \infty$.

Suppose finally that ω is such that

$$S(x) = \sum_{k=1}^{\infty} h(xk)\omega_k$$

exist at every x > 0, and $C'^{(r)}_m(\omega)$ is either (i) convergent as m $\to\infty$, or (ii) finitely oscillating as m $\to\infty$. Then

(7) Under case (i), $\lim_{x\to 0}$ S(x) = (C,r)-sum of ω;

and

(7a) Under case (ii) $\overline{\lim}_{x\to 0}$ S(x) and $\underline{\lim}_{x\to 0}$ S(x) are finite, and S(x) oscillates finitely as x $\to$ 0.

This property of the Cesaro means is well-known and its proof may be found in e.g. Hobson (1921) or elsewhere.

Lemma 5 If ω is (C,r)-bounded, then for every $n>\frac{r+2}{2}$, $\sigma_n(\omega,u)$ is absolutely convergent at each u>0.

Proof Let r$\in$N and $n>\frac{r+2}{2}$. If ω is (C,r)-bounded, then $\omega_k = O(k^r)$, and so we can select a constant b_r such that $(|\omega_k|/k^r)<b_r$ for all k ≥ 1. If now u>0, then from Lemma 1,

$$\sum_{k=0}^{\infty} g_{uk}^{[n]}|\omega_k| < |\omega_0| + \sum_{k=1}^{\infty}\frac{n^{2n-1}|\omega_k|}{(uk)^{2n-1}}$$

$$= |\omega_0| + \frac{n^{2n-1}}{u^{2n-1}}\sum_{k=1}^{\infty}\frac{|\omega_k|}{k^r k^{2n-(r+1)}}$$

$$<|\omega_0| + \frac{n^{2n-1}}{u^{2n-1}} b_r \sum_{k=1}^{\infty} k^{-[2n-(r+1)]},$$

which is finite since 2n-(r+1) > 1 from hypothesis. Thus $\sigma_n(\omega, u)$ is absolutely convergent as asserted.

The inequality obtained in the above proof will be used again, and we identify it by (8) as follows: If r ≥1, and ω is (C,r)-bounded, then for all $n>\frac{r+2}{2}$ and u >0,

(8) $|\sigma_n(\omega,u)| < |\omega_0| + \frac{c_{n,r}}{u^{2n-1}}$

where $c_{n,r}$ > 0 depends only on n, r, and is independent of u.

3. COMPARISON WITH (C,r)

We now match up σ_n with (C,r).

Theorem 1

(a) **[First comparative Efficiency]** For all r ≥ 1 and $n > \frac{r+2}{2}$, the σ_n- summability method is at least as efficient as the (C,r)-mean.

(b) **[Second Comparative Efficiency]** If $n > \frac{r+3}{2}$ then σ_n is more efficient than (C,r).

Thus the summation of divergent series by the σ_n-matrices is more efficient than the (C,r)-mean for almost all n.

[*Note*: A counter-example with $g_{nk}^{|2|} = \frac{2}{(uk+1)^2.(uk+2)}$ and the (C,5)- summable series $\sum_{k=0}^{\infty}(-1)^k(k+1)^4$, explains why we want $n > \frac{r+2}{2}$]

Proof Fix r ≥ 1. Take any $n > \frac{r+2}{2}$. If ω (C,r)-bounded, then by Lemma 5

$$\sigma_n(\omega,u) \equiv \sum_{k=0}^{\infty} g_{uk}^{|n|}\omega_k \text{ exist for } u>0.$$

Writing $h_n(u) = \sigma_n(u)/\sigma_n(0)$, we obtain from Lemma 2,

$$h_n^{(r+1)}(u) = O(u^{-(r+a+1)}) \quad \text{on } u > 1,$$

where a is fixed in 0<a<2n-(r+2). By Lemma 3 we also have $u^r h_n(u) \to 0$ as $u \to \infty$. Furthermore, since h_n is completely monotonic on u > -1, then $h_n \in C^{(r+1)}[0,\infty)$ and h_n is of course continuous at u = 0, with $h_n(0) = 1$. Since ω is (C,r)-bounded, $\overline{\lim_{x\to\infty}} C_m^{(r)}(\omega)$ and $\underline{\lim_{x\to\infty}} C_m^{(r)}(\omega)$ are finite, and the sequence $C_m^{(r)}(\omega)$ oscillates finitely as m → ∞, unless it is convergent. If, in particular, ω is (C,r)-summable, then $C_m^{(r)}(\omega)$ is convergent as m → ∞. Thus all the conditions of Lemma 4 are satisfied by h = h_n for any n >(r + 2)/2. Hence (7) and (7a) apply to.

$$S(u) \equiv \sum_{k=0}^{\infty} h_n(uk)\omega_k = \sum_{k=0}^{\infty} g_{uk}^{|n|}\omega_k = \sigma_n(\omega,u)$$

By (7), the σ_n-summability will be consistent with (C,r) when n > (r+2)/2. By (7a), $\sigma_n(\omega,u)$ is bounded near the origin (from the right). Pick a > 0 such that:

i. On 0 < u ≤a, $\sigma_n(\omega,u)$ is bounded. Applying (8) to the interval u ≥ a, we obtain:

ii. $|\sigma_n(\omega,u)| \leq |\omega_0| + {c_{n,r}}/{a^{2n-1}}$. By (i) and (ii), $\sigma_n(\omega,u)$ is bounded on all of 0<u<∞.

This proves (a).

To prove (b), we will show instead that if r>1 and n>(r+2)/2 then σ_n is more efficient than (C,r-1). For if in the latter we replace r by r+1, we obtain (b). So, assume r >1 and n > (r+2)/2. Then by Theorem 1(a) the σ_n-method is consistent with (C,r), and hence with (C,r-1). But since (C,r) can sum a series not summable (C,r-1), so can σ_n. Hence σ_n is strictly stronger than (C,r-1). It remains to show that whenever ω is (C,r-1)-bounded, it is σ_n-bounded. But by part (a), since we assumed above that n > (r+2)/2>[(r-1) + 2]/2, then σ_n is at least as efficient as (C,r-1). Hence $\sigma_n(\omega,u)$ is bounded as desired. This completes the proof of (b) and the theorem.

Remark The first method of summability related to $J_\nu(z)$, given by Cooke (1937), is of the "(C,1)-scope" only. The methods of this article are of every (C,r)-scope, for all r≥1.

ACKNOWLEDGEMENT

This work is part of a research project carried out under University of Nigeria Grant No. 94/111. I thank also the National Mathematical Centre ABUJA for making its research facilities available.

REFERENCES

Carlitz, L., 1963. A sequence of integers related to Bessel functions; Proc. AMS. 14, pp. 1-9;

Cooke, R. G., 1937. New method for summability of divergent sequences; J.Lond. Math. Soc. 12: 299-304

Cooke, R. G., 1950. Infinite matrices and sequence spaces; McMillan and Co. Ltd., London, P.92.

...bson, E. W. 1921. Theory of functions of a real variable and the theory of Fourier series, Vol. II; Cambridge University Press.

Kishore, N., 1964. A structure of the Rayleigh polynomial; Duke M.J. 31(3): 513-8

Kishore, N., 1967. A class of formulas for the Rayleigh function; Duke M.J. 34 (3): 573-80.

Obi, E. C., 1975. Functional bounds, series and Log convexity of Rayleigh higher derivatives; J. Math. Anal. And Applic. 52(3): 648-59.

Obi, E. C., 1978. A note on the Rayleigh polynomials; SIAM J. Math. Anal. 9(5): 825-34.

Obi, E. C., 1980 . Complete monotonicity of the Rayleigh functions: ibid 77(2): 465-8.

Obi, E. C., 1983. Some Local Properties of the Rayleigh function; Bull. Cal. Math. Soc. 75: 49-55.

Obi, E. C., 1986. The summability significance of the roots of Bessel functions; SIAM J.Math. Anal. 17(6): 1489-94.

Obi, E. C., 1987. Powerful and delicate connections of $J_\nu(z)$ in summability; J. Nig. M. Soc. 6: 33 - 9.

Obi, E. C., 1988. New property of Weierstrass Primary Factors, Indian J. Math. 30(2): 151.