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THE COMPUTATION OF SYSTEM MATRICES FOR BIQUADRATIC SQUARE FINITE ELEMENTS

GABRIEL ATAH OLAYI

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ABSTRACT

An algorithm for the automatic generation of system matrices for biquadratic square elements is established . It is shown that no matter how large the system matrices may be, the mass matrix has only 13 distinct elements while the stiffness matrix has 16. Also established is an algorithm for generating larger system matrices from given system matrices.

KEY WORDS: Biquadratic Square Elements, System Matrices, Finite Element Method.

INTRODUCTION

Finite Element Method is the most powerful numerical technique for solving partial differential equation problems (Strang and Fix, 1973 p1-135; Olayi 2001). The purpose of this paper is to produce the distinct entries of the mass and stiffness matrices and so generate the system matrices associated with some finite element approximations for the solution of self-adjoint, second order partial differential equations using square elements. It is hoped that this result together with others in the series (Olayi, 2001; Olayi, 2000) will encourage others to develop algorithms for automatic generation of system matrices of other types of finite elements. In this way, we will end up with algorithms and tables for the distinct entries of system matrices corresponding to different types of finite elements as it has been done for the generation of finite elements (Bellingeri, etc, 1995). This will not only reduce the initial amount of work involved in using the Finite Element Method , but will greatly reduce the complexity in Finite Element Programming, and open up the method for more applications and to more users.

A relationship is also established between the distinct entries of system matrices for two separate subdivisions of a square domain with the same bi-k^{th} finite elements.

DEFINITIONS

A Biquadratic Element is a polynomial of degree two in each of two variables and possessing $(2+1)^2$ degrees of freedom within the element. More generally, a Bi-k^{th} element is a polynomial of degree k in each of two variables and possessing $(k+1)^2$ degrees of freedom within the element. For the construction of these elements see (Olayi, 1977) and (Strang and Fix, 1973).

An element basis function is a polynomial which is one at the node with which it is associated and zero at all other nodes of the element. A basis function φ_j, for a trial space is a piecewise polynomial whose pieces are the element basis functions associated with the j^{th} node of the system.

The matrices $M = (m_{ij})$, $m_{ij} = \iint_R \phi_i \phi_j dxdy$ and

$K = (k_{ij}), k_{ij} = \iint_R \left[\phi_{i,x}\phi_{j,x} + \phi_{i,y}\phi_{j,y}\right] dxdy$ where $\phi_{\ell,t} = \frac{\partial}{\partial t}\phi_\ell$ and the ϕ_ℓ are the basis functions

for trial space on R, are called Finite Element System Matrices (Norrie & de Vries, 1973) or when

GABRIEL ATAH OLAYI: Department of Mathematics, University of Calabar, Calabar, Nigeria

there is no ambiguity, System matrices. M is called the mass matrix and K is called stiffness matrix (Barry, 1974; Norrie & de Vries, 1973; Strang & Fix, 1973).

The square, $s = \{(x, y) \mid 0 \le x, y \le h\}$ will be called the standard square element or standard element when there is no ambiguity.

THE BIQUADRATIC SYSTEM MATRICES

Theorem 1 gives the value of the distinct entries in the M and K matrices for a system of biquadratic square elements. The restriction of the domain to a square does not restrict the applicability of the theorem (Olayi, 2000). On how to construct the basis functions for the trial space consult any of the references especially one of Norrie & de Vries, 1973; Olayi, 1977; Strang & Fix, 1973 and Strang, 1972. The proof of this theorem gives the algorithm by which the full M and K matrices can be generated.

Theorem 1. A square R is divided into N^2 squares of size h. If $\varphi_j(x, y)$, $j = 1, 2, ..., n^2$ where $n = 2N + 1$, are the basis functions of the trial space composed of biquadratic square elements then

(a) $$\left\{ \iint_R \phi_i \phi_j dxdy \mid i, j = 1,2,\ldots,n^2 \right\}$$

$= \{-16m_2, -8m_2, -4m_2, -2m_2, 0, m_2, 4m_2, 8m_2, 16m_2, 32m_2, 64m_2, 128m_2, 256m_2\}$

(b) $$\left\{ \iint_R \left[\phi_{i,x}\phi_{j,x} + \phi_{i,y}\phi_{j,y}\right] dxdy \mid i, j = 1,2,\ldots,n^2 \right\}$$

$= \{-112k_2, -96k_2, -35k_2, -32k_2, -18k_2, -6k_2, -2k_2, 0, 10k_2, 42k_2, 56k_2, 112k_2, 176k_2, 224k_2, 352k_2, 512k_2\}$

where $m_2 = h^2/900$, $k_2 = 1/90$.

Proof: The value $m_{ij} = \iint_R \phi_i \phi_j dxdy$ and $k_{ij} = \iint_R \left[\phi_{i,x}\phi_{j,x} + \phi_{i,y}\phi_{j,y}\right] dxdy$ each depends on the relative positions of nodes i and j. These fall into 17 classes and produce 13 distinct values for m_{ij} and 16 for k_{ij} as follows:

D. $i = j$

D-1. If j is a corner node of R then $m_{jj} = 16m_2$, $k_{jj} = 56k_2$.

D-2. If j is a midedge node (i.e. node at the midpoint of an edge) on the boundary of R then m_{jj} $64m_2$, $k_{jj} = 176k_2$.

D-3. If j is a boundary node other than those of D-1 and D-2 then $m_{jj} = 32m_2$, $k_{jj} = 112k_2$.

D-4. If j is a node at the centre of an element then, $m_{jj} = 256m_2$, $k_{jj} = 512k_2$.

D-5. If j is a midedge node in the interior of R then $m_{jj} = 128m_2$, $k_{jj} = 352k_2$.

D-6. If j is a interior node other than those D-4 and D-5 then $m_{jj} = 64m_2$, $k_{jj} = 224k_2$.

O. $i \ne j$

O-1. If i and j are adjacent nodes on the boundary of R then $m_{ij} = 8m_2$, $k_{ij} = -18k_2$.

O-2. If i and j are at distance h apart, on the same edge and on the boundary of R, then $m_{ij} = -4m_2$, $k_{ij} = -35k_2$.

O-3. If i and j are adjacent nodes, at least one of which is in the interior of R, then $m_{ij} = 16m_2$, $k_{ij} = -42k_2$.

O-4. If one of i, j is a node at a vertex of an element and the other a centre node of the same element, or if i and j are midedge nodes of edges that meet at a vertex of the same element, then $m_{ij} = 4m_2$, $k_{ij} = -32k_2$.

O-5. If i and j are nodes at the two ends of a diagonal of an element, then $m_{ij} = m_2$, $k_{ij} = -2k_2$.

O-6. If one of i, j is a centre node and the other a midedge node of the same element, then $m_{ij} = 32m_2$, $k_{ij} = -96k_2$.

O-7. If one of i, j is a midedge node and the other a vertex node on a different edge of the same element, then $m_{ij} = -2m_2$, $k_{ij} = 10k_2$.

O-8. If i and j are midedge nodes of parallel edges of the same element then $m_{ij} = -16m_2$, $k_{ij} = 0$.

O-9. If i and j are not nodes of the same element then $m_{ij} = 0 = k_{ij}$.

O-10. If i and j are h units apart on the same edge and one of i or j is in the interior of R, then $m_{ij} = -8m_2$, $k_{ij} = -6k_2$.

O-11. If i and j are h units apart on the same edge with both i and j in the interior of R, then $m_{ij} = -8m_2$, $k_{ij} = 42k_2$.

If each square element is transformed into the standard element, then the product of only the following integrals are involved in the computation of m_{ij} and k_{ij} for biquadratic square finite elements.

(i) $\int_0^h w^2(w/h-\tfrac{1}{2})(1-w/h)dw = \tfrac{h^3}{120}$.

(ii) $\int_0^h w^2(w/h-\tfrac{1}{2})^2 dw = \tfrac{h^3}{30} = \int_0^h w^2(w/h-1)^2 dw$

(iii) $\int_0^h w(w/h-\tfrac{1}{2})^2(w/h-1)dw = -\tfrac{h^2}{120}$.

(iv) $\int_0^h w(w/h-\tfrac{1}{2})^2(w/h-1)^2 dw = \tfrac{h}{30}$.

(v) $\int_0^h (1-2wh)^2 dw = \tfrac{h}{3}$.

(vi) $\int_0^h (1-2w/h)(2w/h-\tfrac{3}{2})dw = -\tfrac{h}{3} = \int_0^h \left(1-\tfrac{2w}{h}\right)\left(\tfrac{2w}{h}-\tfrac{1}{2}\right)dw$.

(vii) $\int_0^h \left(\tfrac{2w}{h}-\tfrac{1}{2}\right)^2 dw = \tfrac{7h}{12} = \int_0^h \left(2w/h-\tfrac{3}{2}\right)^2 dw$.

(viii) $\int_0^h \left(2w/h-\tfrac{3}{2}\right)\left(2w/h-\tfrac{1}{2}\right)dw = \tfrac{h}{12}$.

RELATION BETWEEN SYSTEM MATRICES

Theorem 2. A square R is divided into N_1^2 squares of size $h_1 \cdot M^{(1)}$ and $K^{(1)}$ are the corresponding system matrices for a system of bi-kth elements on R. If R were subdivided into N_2^2 square of size h_2 with $M^{(2)}, K^{(2)}$ the corresponding system matrices for a system of bi-kth elements, then

(1) $m_j^{(2)} = \frac{1}{\beta^2} m_j^{(1)}, j = 1,2,\ldots,J$

(2) $k_\ell^{(2)} = k_\ell^{(1)}, \ell = 1,2,\ldots,L$

where $\beta = \frac{h_1}{h_2}$, J and L are the total distinct number of entries in the mass and stiffness matrices for a system of bi-kth elements. $m_j^{(1)}, m_j^{(2)}$ are the corresponding distinct entries in $M^{(1)}$ and $M^{(2)}$, with $k_\ell^{(1)}$ and $k_\ell^{(2)}$ similarly defined.

Proof: For each of the two subdivisions, each elements is transformed into standard element s_1, s_2.

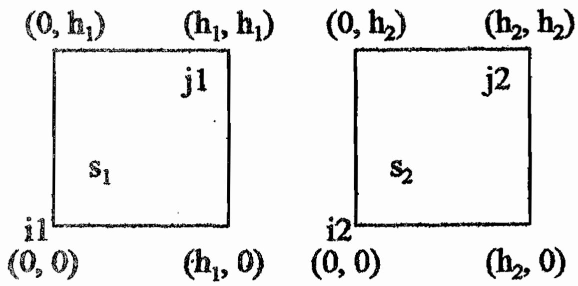

Fig 2. Standard elements in the two subdivisions.

Let ϕ_{i1}, ϕ_{j1} be the basis function for node i1, j1 respectively of s_1 and ϕ_{i2}, ϕ_{j2} the basis functions for corresponding nodes i2, j2 of s_2.

$$\phi_{i2}(t,u) = \phi_{i1}\left(\frac{h_1 t}{h_2}, \frac{h_1 t}{h_2}\right)$$

$$= \phi_{i1}(\beta t, \beta u) \qquad \text{(i)}$$

Similarly $\phi_{j2}(t, u) = \phi_{j1}(\beta t, \beta u)$ (ii)

$$m_{i2,j2}^{(2)} = \int_0^{h_2}\int_0^{h_2} \phi_{i2}(t,u)\phi_{j2}(t,u)dtdu$$

$$= \int_0^{h_2}\int_0^{h_2} \phi_{i1}(\beta t,\beta u)\phi_{j1}(\beta t,\beta u)dtdu \text{ from (i), (ii)}$$

$$= \frac{h_2^2}{h_1^2}\int_0^{h_1}\int_0^{h_1} \phi_{i1}(v,w)\phi_{j1}(v,w)dvdw,$$

$$= \frac{1}{\beta^2} m_{i1,j1}^{(1)}.$$

And

$$k_{i2,j2}^{(2)} = \int_0^{h_2}\int_0^{h_2}\left[\ \phi_{i2,t}(t,u)\phi_{j2,t}(t,u) + \phi_{i2,u}(t,u)\phi_{j2,u}(t,u)\ \right]dtdu$$

$$= \beta^2\int_0^{h_2}\int_0^{h_2}\left[\ \phi_{i1,t}(\beta t,\beta u)\phi_{j1,t}(\beta \xi,\beta u) + \phi_{i1,u}(\beta t,\beta u)\phi_{j1,u}(\beta t,\beta u)\ \right]dtdu$$

$$= \int_{S_1}\int \left[\phi_{I1,v}(v,w)\phi_{J1,v}(v,w) + \phi_{I1,w}(v,w)\phi_{J1,w}(v,w) \right] dvdw,$$

$$= k^{(1)}_{I1,J1}.$$

This completes the proof of theorem 2.

Note: Theorem 2 is a generalized version of lemma 5.1 in (Olayi, 1977) and theorem 2 in (Olayi, 2000). For bilinear elements J = 6 = L; for biquadratic elements J = 13 and L = 16. For general bi-kth elements J and L are not exactly known but can be determined once k is known and are usually less than half the dimension of the larger trial space.

CONCLUSION

The proof of Theorem 1 provides an algorithm for the automatic generation of system matrices for biquadratic square elements and shows that no matter how large the system matrices may be, the mass matrix contains only 13 distinct elements while the stiffness matrix has 16. Theorem 2 gives an algorithm for generating larger system matrices from given system matrices and shows that the values of the distinct elements of stiffness matrix remain unchanged.

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