GLOBAL JOURNAL OF PURE AND APPLIED SCIENCES VOL. 7, NO. 2, MARCH 2001:389-396
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SOME OPTIMALITY CONDITIONS FOR THE EXISTENCE OF OPTIMIZERS OF CERTAIN CLASS OF LINEAR PROGRAMMING PROBLEMS

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(Received 16 December 1999; Revision accepted 6 October 2000)

Abstract

It is well known that the optimum of a Linear Programming problem occurs at an extremum point of the feasible region. This paper considers some other optimality conditions for the existence of optimizers of Linear Programming (LP) problems based on the principles of optimal experimental design. It is shown in this work, for example, that;

(i) The optimizer of an LP Problem occurs at a point, which the d-function is minimum within the feasible region.

(ii) The d-function at the end point of the kth iteration is less than the d-function at any other point within the experimental space.

(iii) The d-function at the minimizer $\underline{x}^*$ is the maximum of the minimum d-functions for k different iterations. This is a first-order necessary condition for the existence of a minimizer of an LP problem.

Key words: Linear Programming, d-function, Linear Exchange Algorithm, Optimal Experimental Design.

Introduction

Linear programming (LP) problems are characterised by a linear objective function in n non-negative variables constrained by a set of m (usually, m<n) linearly independent equations. These constraints which may be linear equality or linear inequality define a convex feasible region.

We remark that since the objective function of an LP problem is non-stochastic, we shall throughout this work define, $\underline{x}'M^{-1}(\xi_n)$, $\underline{x}\in S_x$ to be the d-function rather than variance of the objective function; S_x is the column space of the information matrix at the kth iteration; i.e $d(f(\underline{x})) = d(\underline{x},\xi_n) = \underline{x}'M^{-1}(\xi_n)\underline{x}$, $\underline{x}\in S_x \subset \tilde{X}$ where ξ_n is the design measure of the exact design and X is the feasible region.

We further remark that in developing the optimality conditions, some restriction has been placed on the objective function $f(\underline{x})$; ie $f(\underline{x}) \geq 0$, $\underline{x}\in \tilde{X}$. Thus, LP problems for which $f(\underline{x})$ can change sign from positive to negative or from negative to positive within the feasible region are not covered here.

Each of the optimality conditions discussed in this paper is based on the Linear Exchange Algorithm (LEA), itself a line search algorithm, for solving Linear Programming Problems developed by making use of the principles of experimental design. The basic steps involved in the LEA are given as follows:

S_1 : at the boundary of $\tilde{X}$, take n_0 support points for the initial design matrix

$$X_{0k} = (\underline{x}_{1k}, \underline{x}_{2k}, \ldots\ldots, \underline{x}_{mk} \ldots\ldots, \underline{x}_{n0k})'$$

such that $\det(X_{0k}'X_{0k}) \neq 0$ and $n + 1 \leq n_0 \leq \frac{1}{2}n\,(n+1)$

S_2 : make a move in the gradient direction to the point

$$\underline{x}_k = \underline{x}_{0k} - \alpha_{0k}\underline{g}, \quad k = 1, 2, \ldots ,; \quad \bar{\underline{x}}_{0k} = X_{0k}\,1/n_0$$

$$\underline{g} = \left[\frac{\partial f(\underline{x})}{\partial x_i}\right]', \quad i = 1, 2, \ldots, n$$

$$\alpha_{0k} = \min_i \alpha_j = \min_i \left\{\frac{a_i'\bar{\underline{x}}_{0k} - b_i}{a_i'\underline{g}}\right\}, \quad j = 1, \ldots, m$$

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S_3: if $\underline{x}_k = \underline{x}^*$, the minimizer, stop otherwise replace $\underline{x}_{mk}$ in X_{ok} with $\underline{x}_k$; i.e define
$X_{OK(k+1)} = (\underline{x}_{1k}, \underline{x}_{2k}, \ldots, \underline{x}_{mk}, \ldots, \underline{x}_{nOk})'$
and $\bar{\underline{x}}_{o(k+1)} = X'_{o(k+1)}\underline{1}/n_o$, where $\underline{x}_{mk}$ is such that $f(\underline{x}_{mk}) > f(x_{ik})$, $i = 1, 2, \ldots, n_o$; $i \neq m$.

S_4: set $k+1 = k$ and return to step S_2.

The sequence terminates whenever

$$\frac{|f(x_{k+1}) - f(x_k)|}{|f(\underline{x}_k)|} < \varepsilon, \ \varepsilon > 0$$

or $$|\underline{x}_{Ok} M_{Ok}^{-1} \underline{x}_{Ok} - \underline{x}_k' M_k^{-1} \underline{x}_k| < \delta, \ \delta > 0;$$

M_k is the information matrix at the kth iteration. That is, the sequence terminates whenever the difference between the d-function at the starting point $\bar{\underline{x}}_{Ok}$ and the d-function at $\underline{x}_k$, the end point of the kth iteration is small; i.e. no significant move is made from $\bar{\underline{x}}_{Ok}$ to $\underline{x}_k$. A detailed discussion on the operation of this algorithm is given in Umoren (2000).

In developing the optimality conditions, we require the following lemma that is useful in obtaining the inverses and determinants of information matrices for line search algorithms.

Lemma 1.1: Let $B = A + \underline{u}\underline{v}'$ be an $m \times m$ matrix such that $\det(A) \neq 0$, $\underline{u}$ and $\underline{v}$ are m component non-zero vectors, then

$$B^{-1} = A^{-1} - A^{-1}\underline{u}'(1 + \underline{v}'A^{-1}\underline{u})^{-1}\underline{v}A^{-1} \qquad (1.1.1)$$

and

$$\det(B) = \det A(1 + \underline{v}'A^{-1}\underline{u}) \qquad (1.1.2)$$

The above lemma in linear algebra has been referred to or proved by many authors including Rao (1965), Raghavarao (1971), hence the proof is here omitted.

Lemma 1.2: Given the line search equation

$$\underline{x}_{j+1} = \underline{x}_j - \rho_j \underline{d}_j; \ \underline{d}_j \underline{d}_j' = 1$$

Where $\underline{d}_j$, and ρ_j are respectively the direction of search and step length at the jth iteration.

Let $M_j, M_{j+1} \in M^{nxn}$ be information matrices at the jth and (j+1)th iteration respectively, M^{nxn} is the set of all information matrices. Then

(i) $\det(M_{j+1}) = \det(M_j)(1 + n_o\rho_o^2\underline{d}_j'M_j^{-1}\underline{d}_j)$ or

$$\frac{\det(M_{j+1})}{\det(M_j)} = 1 + w_j; \ w_j = n_o\rho_o^2\underline{d}_j'M_j^{-1}\underline{d}_j$$

(ii) $M_{j+1}^{-1} = M_{j+1}^{-1} - \underline{z}_j\underline{z}_j$; $\underline{z}_j = n_o^{1/2}\rho_j(1 + w_j)^{1/2}M_j^{-1}\underline{d}_j$

Proof: (i) At $\underline{x}_j$, the column space of the design matrix X_j, ($n_o \times n$) is spanned by the vector $\underline{x}_j = (x_{j1}, x_{j2}, \ldots, x_{jn})$; i.e $\underline{x}_j \in L(X_j)$. Similarly at $\underline{x}_{(j+1)}$ the column space of the design matrix X_{j+1}, ($n_o \times n$) is spanned by the vector $\underline{x}_{j+1} = (x_{j+1,1}, x_{j+2,2}, \ldots, x_{j+1,n})$. Therefore

$$\underline{x}_{j+1} = (x_{j1} - \rho_{j1}\underline{d}_{j1}, x_{j2} - \rho_{j2}\underline{d}_{j2}, \ldots, x_{jn} - \rho_{jn}\underline{d}_{jn}) \quad \text{and}$$
$$X_{j+1} = X_j - \rho_j'\underline{1}\underline{d}_j; \quad \underline{d}_j' = (d_{j1}, \ldots, d_{jn})$$
$$X_{j+1}'X_{j+1} = X_j'X_j - \rho_jX_j'\underline{1}\underline{d}_j - \rho_j d_j \underline{1}'X_j + \rho^2 n_o\underline{d}_j\underline{d}_j'$$
$$= X_j'X_j + n_o\rho_j^2\underline{d}_j\underline{d}_j'; \quad X_j'\underline{1} = 0$$
$$\text{i.e} \quad M_{j+1} = M_j + n_o\rho_j^2\underline{d}_j\underline{d}_j' \qquad (1.2.1)$$

and

$$\det(M_{j+1}) = \det(M_j)(1 + n_o\rho_j^2\underline{d}_j'M_j^{-1}\underline{d}_j)$$

or

$$\frac{\det(M_{j+1})}{\det(M_j)} = 1 + w_j; \quad w_j = n_o\rho_j^2\underline{d}_j'M_j^{-1}\underline{d}_j$$

(ii) On application of (1.1.2) in (1.2.1) we have

$$M_{j+1}^{-1} = M_j^{-1} - \frac{M_j^{-1}n_o\rho_j^2\underline{d}_j\underline{d}_j'M_j^{-1}}{1 + n_o\rho_j^2\underline{d}_j'M_j^{-1}\underline{d}_j}$$

$$= M_j^{-1} - M_j^{-1} n_o p_j^2 \underline{d}_j{}'(1 + n_o p_j^2 \underline{d}_j{}' M_j^{-1} \underline{d}_j)^{-1} \underline{d}_j M_j^{-1}$$
$$= M_j^{-1} - [n_o^{1/2} p_j(1+w_j)^{1/2} M_j^{-1} \underline{d}_j][n_o^{1/2} p_j(1+w_j)^{1/2} M_j^{-1} \underline{d}_j]$$
$$= M_j^{-1} - z_j z_j{}' \;;\quad z_j = n_o^{1/2} p_j(1+w_j)^{1/2} M_j^{-1} \underline{d}_j$$

Definition 1.3: Let $f(\underline{x})$ be the objective function of an LP problem and let $\underline{x}^*$ be the optimizer (maximizer or minimizer), $\underline{x} \in \tilde{X}$. Then

$$f(\underline{x}^*) \begin{cases} \geq f(\underline{x}), & \text{if } f(\underline{x}) \text{ is to be maximized} \\ \leq f(\underline{x}), & \text{if } f(\underline{x}) \text{ is to be minimized} \end{cases}$$

Pazman (1986) has established a functional relationship between the square of a response function and the variance of the function. That is, the variance of the BLUE for a linear functional h defined on Ω is

$$\begin{aligned} Var(\hat{h}) &= v_h M \underline{v}_h = \underline{u}' M^{-1} \underline{u} \\ &= \max\left\{ \frac{(\underline{u}'\underline{a})^2}{\underline{a}' M \underline{a}},\ \underline{a} \in R^n,\ M\underline{a} \neq 0 \right\} \quad (1.1.4) \\ &= \max\left\{ 2\underline{u}'\underline{a} - \underline{a}' M \underline{a},\ \underline{a} \in R^n \right\} \quad (1.1.5), \end{aligned}$$

if $h(w) = \underline{u}'\underline{a}$, $w \in \Omega$, where $\underline{v}_h \in L(M)$ and $\underline{u} = M\underline{v}_h \Rightarrow \underline{v}_h = M^{-1}\underline{u}$.

Correspondingly, since the objective function of an LP problem, namely $f(\underline{x}) = \underline{c}'\underline{x}$ is a linear functional, the relationship between its square and its d-function can be derived from (1.1.4) and (1.1.5) as

$$\begin{aligned} df(\underline{x}) = \underline{x}' M^{-1} \underline{x} &= \max\left\{ \frac{(\underline{c}'\underline{x})^2}{\underline{c}' M \underline{c}},\ \underline{c} \in R^n,\ M\underline{c} \neq 0 \right\} \\ &= \max\left\{ \frac{f^2(\underline{x})}{\underline{c}' M \underline{c}},\ \underline{c} \in R^n,\ M\underline{c} \neq 0 \right\} \quad (1.1.6) \\ &= \max\left\{ 2\underline{c}'\underline{x} - \underline{c}' M \underline{c},\ \underline{c} \in R^n \right\} \quad (1.1.7) \end{aligned}$$

2.1 Optimality Conditions

It is well known from the point of view of mathematics that, at the optimizer $\underline{x}^*$, the gradient of the objective function vanishes; i.e

$$\frac{\partial f(\underline{x})}{\partial x_i} = 0, \quad I = 1, 2, \ldots, n$$

From definition (1.3), if $\underline{x}^*$ is a minimizer and $f(\underline{x}) \geq 0$, then

$$f(\underline{x}^*) \leq f(\underline{x}) \Leftrightarrow f^2(\underline{x}^*) \geq f^2(\underline{x}) \quad (2.1.1)$$

Let us now lay a foundation for the development of the optimality conditions. We first show that in the LEA, the sequence moves from a point of relatively high d-function to a point of lower d-function.

Theorem 2.2: Given the line search sequence

$$\underline{x}_k = \bar{\underline{x}}_{ok} - \alpha_{ok}\underline{g}; \quad \bar{\underline{x}}_{ok} = X_{ok}{}'\underline{1}/n_o$$

Then

$$\underline{x}_k{}' M_k^{-1} \underline{x}_k \leq \bar{\underline{x}}_{ok}{}' M_k^{-1} \bar{\underline{x}}_{ok}, \quad \underline{x}_k \in S_x$$

Where $\underline{x}_k$ and M_k are the end points and information matrix at the kth iteration and S_x is the column space of the information matrix at the kth iteration.

Proof: $\underline{x}_k = \bar{\underline{x}}_{ok} - \alpha_{ok}\underline{g}$

$$\begin{aligned} \Rightarrow \underline{x}_k{}' M_k^{-1} \underline{x}_k &= (\bar{\underline{x}}_{ok} - \alpha_{ok}\underline{g})' M_k^{-1} (\bar{\underline{x}}_{ok} - \alpha_{ok}\underline{g}) \\ &= \bar{\underline{x}}_{ok}{}' M_k^{-1} \bar{\underline{x}}_{ok} - 2\alpha_{ok} \bar{\underline{x}}_{ok} M_k^{-1} \underline{g} + \alpha_{ok}^2 \underline{g}' M_k^{-1} \underline{g} \\ &= \bar{\underline{x}}_{ok}{}' M_k^{-1} \bar{\underline{x}}_{ok} - 2\underline{c}' \bar{\underline{x}}_{ok} + \underline{c}' M_k \underline{c} \end{aligned}$$

on setting $\alpha_{ok}M_k^{-1}g = \underline{c}$, the coefficient of the objective function.

R.H.S. $\Rightarrow \bar{x}_{ok}'M_k^{-1}\bar{x}_{ok} \geq 2\underline{c}'\bar{x}_{ok} - \underline{c}'M_k\underline{c}$

$\Rightarrow \bar{x}_{ok}'M_k^{-1}\bar{x}_{ok} = \max\{2\underline{c}'\bar{x}_{ok} - \underline{c}'M_k\underline{c}\} \geq 0$

from (1.1.5). Therefore,

$$\underline{x}_k'M_k^{-1}\underline{x}_k \leq \bar{x}_{ok}'M_k^{-1}\bar{x}_{ok}$$

We now show that if the starting point of the sequence is a weighted average, then the value of the objective function at the kth iteration is less than its value for any other $\underline{x} \in S_x \subset \tilde{X}$. But before that let us state the following lemma that is fundamental to the proof of the theorem that follows.

Lemma 2.3: Given

$$\underline{x}_i'M_k^{-1}\underline{x}_i \leq \underline{x}_j'M_k^{-1}\underline{x}_j, \quad \underline{x}_i, \underline{x}_j \in S_x$$

Then

$$f^2(\underline{x}_i) \leq f^2(\underline{x}_j) \Leftrightarrow f(\underline{x}_i) \leq f(\underline{x}_j)$$

where M_k is the information matrix at the kth iteration.

Proof: $\underline{x}_i'M_k^{-1}\underline{x}_i \leq \underline{x}_j'M_k^{-1}\underline{x}_j$

$\Rightarrow tr(M_k^{-1}\underline{x}_i\underline{x}_i') \leq tr(M_k^{-1}\underline{x}_j\underline{x}_j')$

$\Rightarrow \underline{x}_i\underline{x}_i' \leq \underline{x}_j\underline{x}_j'$

$\Rightarrow f^2(\underline{x}_i) \leq f^2(\underline{x}_j) \Leftrightarrow f(\underline{x}_i) \leq f(\underline{x}_j)$

Theorem 2.3: Given the line search sequence

$\underline{x}_k = \bar{x}_{ok} - \alpha_{ok}g \quad ; \quad \bar{x}_{ok} = \sum\alpha_i\underline{x}_i, \ \alpha_i \geq 0; \ \sum\alpha_i = 1$

and $\underline{x}_i$'s are independent. Then

$f(\underline{x}_k) \leq f(\underline{x}), \ \underline{x} \in S_x$

Proof: The proof is by induction

Set $n_o = 2$

$\bar{x}_{1ok} = \alpha x_1 + (1 - \alpha)x_2, \quad \alpha \geq 0$

From Theorem 2.2

$d_k = x_k'M_k^{-1}\underline{x}_k \leq d_{1ok} = \bar{x}_{ok}'M_k^{-1}\bar{x}_{ok}$

$d_{1ok} = \alpha^2 x_1'M_k^{-1}\underline{x}_1 + (1 - \alpha)^2 \underline{x}_2'M_k^{-1}\underline{x}_2$

On setting $\underline{x}_i'M_k^{-1}\underline{x}_i = d_i, \quad i = 1, 2$, we have

$d_{ok} = \alpha^2 d_1 + (1 - \alpha)^2 d_2$

Choose α to minimize d_{ok}; i.e.

$$\frac{\partial d_{1ok}}{\partial\alpha} = 2\alpha d_1 + 2(1 - \alpha)d_2 = 0$$

$$\alpha = \frac{d_2}{d_1 + d_2}$$

Therefore

$$d_{1ok} = \left(\frac{d_2}{d_1 + d_2}\right)^2 d_1 + \left(\frac{d_1}{d_1 + d_2}\right)^2 d_2$$

$$= \frac{d_1 d_2 \ (d_1 + d_2)}{(d_1 + d_2)^2} = \frac{d_1 d_2}{d_1 + d_2} \leq d_1, d_2$$

Hence $d_k \leq d_1, d_2$

From lemma 2.3 the above inequality implies

$f(\underline{x}_k) \leq f(\underline{x}_1), f(\underline{x}_2)$

Set $n_o = 3$

Define $\underline{x}_{12} \neq \alpha\underline{x}_1 + (1 - \alpha)\underline{x}_2$, and $\bar{x}_{20k} = \beta\bar{x}_{12} + (1 - \beta)\underline{x}_3, \ \beta \geq 0$.

Then

$d_{20k} = \beta^2\bar{d}_{12} + (1 - \beta)^2 d_3$

The value of β which minimizes d_{20k} is $\beta = \dfrac{d_3}{\bar{d}_{12} + d_3}$. Therefore

$$d_{20k} = \left(\frac{d_3}{\bar{d}_{12} + d_3}\right)^2 \bar{d}_{12} + \left(\frac{\bar{d}_{12}}{\bar{d}_{12} + d_3}\right)^2 d_3$$

$$= \frac{\bar{d}_{12} d_3}{(\bar{d}_{12} + d_3)^2} (d_3 + \bar{d}_{12})$$

$$= \frac{\bar{d}_{12} d_3}{\bar{d}_{12} + d_3} \leq \bar{d}_{12}, d_3$$

But $d_{20k} \leq \bar{d}_{12} = \alpha^2 d_1 + (1-\alpha)^2 d_2 \leq d_1, d_2$
Hence,

$$d_{20k} \leq d_1, d_2, d_3 \Rightarrow d_k \leq d_1, d_2, d_3$$

From lemma 2.3, the above inequality implies

$$f(\underline{x}_k) \leq f(\underline{x}_1), f(\underline{x}_2), f(\underline{x}_3)$$

Hence, by induction

$$f(\underline{x}_k) \leq f(\underline{x}_1), f(\underline{x}_2), \ldots, f(\underline{x}_{n0})$$

We now show that if the value of the objective function is minimum at $\underline{x}_k$ within the experimental space S_x, then the d-funcion at $\underline{x}_k$ is minimum for all $\underline{x} \in S_x$.

Theorem 2.4: Given a line search equation

$$\bar{\underline{x}}_k = \bar{\underline{x}}_{ok} - \alpha_{ok}\underline{g} \quad ; \quad \bar{\underline{x}}_{ok} = \sum \alpha_i \underline{x}_i, \quad \alpha_i \geq 0$$

Then

$$\underline{x}_k' M_k^{-1} \underline{x}_k \leq \underline{x}' M_k^{-1} \underline{x}, \quad \underline{x} \in S_x \subset \tilde{X}$$

Proof: Define $f(\underline{x}_k) = \underline{c}'\underline{x}_k \Leftrightarrow f^2(\underline{x}_k) = \underline{x}_k'\underline{c}\underline{c}'\underline{x}_k$ and
$f(\underline{x}) = \underline{c}'\underline{x} \Leftrightarrow f^2(\underline{x}_k) = \underline{x}'\underline{c}\underline{c}'\underline{x}$

From theorem 2.2.

$$\begin{aligned}
f(\underline{x}_k) \leq f(\underline{x}) &\Leftrightarrow f^2(\underline{x}_k) \leq f^2(\underline{x}), \quad x \in S_x \\
&\Rightarrow \underline{x}_k'\underline{c}\underline{c}'\underline{x}_k \leq \underline{x}'\underline{c}\underline{c}'\underline{x} \\
&\Rightarrow \underline{c}'\underline{x}_k\underline{x}_k'\underline{c} \leq \underline{c}'\underline{x}\underline{x}'\underline{c} \\
&\Rightarrow \underline{x}_k\underline{x}_k' \leq \underline{x}\underline{x}' \\
&\Rightarrow tr(\underline{x}_k\underline{x}_k' M_k^{-1}) \leq tr(\underline{x}\underline{x}' M_k^{-1}) \\
&\Rightarrow \underline{x}_k' M_k^{-1} \underline{x}_k \leq \underline{x}' M_k^{-1} \underline{x}, \ \underline{x} \in S_x
\end{aligned}$$

Having in view the functional relationship between the square of the objective function and its d-function (see 1.1.6) we give the following theorem which states that the minimizer of an LP problem occurs at the point where the d-function is minimum within the feasible region.

Theorem 2.5: Given $M_\bullet = \lim_{k\to\infty} M_k$, where $M_\bullet$ is the information matrix at the point of convergence of the sequence. If $\underline{x}^*$ is a minimizer, then

$$\underline{x}^{*\prime} M_\bullet^{-1} \underline{x}^* \leq \underline{x}' M_\bullet^{-1} \underline{x}; \quad \underline{x} \in \tilde{X}$$

and if x^* is a maximizer

$$\underline{x}^{*\prime} M_\bullet^{-1} \underline{x}^* \geq \underline{x}' M_\bullet^{-1} \underline{x}; \quad \underline{x} \in \tilde{X}$$

Proof: Let $f(\underline{x}^*)$ be represented by a linear functional $\underline{x}^{*\prime}\underline{c}$. Then

$$f^2(\underline{x}^*) = \underline{x}^{*\prime}\underline{c}\underline{c}'\underline{x}^*; \quad f^2(\underline{x}) = \underline{x}'\underline{c}\underline{c}'\underline{x}$$

Therefore

$$f^2(\underline{x}^*) \leq f^2(\underline{x}) \Leftrightarrow \underline{x}^{*\prime}\underline{c}\underline{c}'\underline{x}^* \leq \underline{x}'\underline{c}\underline{c}'\underline{x} \qquad \text{(i)}$$

Define $\underline{f}(x) = X\underline{c}$, where X is the design matrix, so that $\underline{c} = (X'X)^{-1}X'\underline{f}(\underline{x})$.

Then (i) becomes

$$\underline{x}^{*\prime}[(X'X)^{-1}X'\underline{f}(\underline{x})\underline{f}'(\underline{x})X(X'X)^{-1}]\underline{x}^* \leq \underline{x}'[(X'X)^{-1}X'\underline{f}(\underline{x})\underline{f}'(\underline{x})X(X'X)^{-1}]\underline{x}$$

$$\Rightarrow \underline{x}^{*\prime}(X'X)^{-1}\underline{x}^* \leq \underline{x}'(X'X)^{-1}\underline{x}$$

$$\Rightarrow \underline{x}^{*\prime}M^{-1}\underline{x}^* \leq \underline{x}'M^{-1}\underline{x}$$

$$\Rightarrow \underline{x}^{*\prime}M_{\bullet}^{-1}\underline{x}^* \leq \underline{x}'M_{\bullet}^{-1}\underline{x}, \ \underline{x} \in \tilde{X}$$

Similarly, $f^2(x^*) \geq f^2(\underline{x})$

$$\Rightarrow \underline{x}^{*\prime}M_{\bullet}^{-1}\underline{x}^* \geq \underline{x}'M_{\bullet}^{-1}\underline{x}, \ \underline{x} \in \tilde{X}$$

We now state a first order necessary condition for the existence of optimizers of LP problems. That is, we show that if the d-function at the end point of the kth iteration is less than the d-function at any other point within the experimental space, then the d-function at the minimizer $\underline{x}^*$ is the maximum of the minimum d-functions at k different iterations. But before that, let us state a lemma that is fundamental to the proof of the theorem that follows:

Lemma 2.6: Given the line search sequences

$$\underline{x}_k = \bar{\underline{x}}_{ok} - \alpha_{ok}\underline{g}, \ \bar{\underline{x}}_{ok} = X_{ok}'\underline{1}/n_o$$

and

$$\underline{x}_{k+1} = \bar{\underline{x}}_{o(k+1)} - \alpha_{o(k+1)}\underline{g}, \ \bar{\underline{x}}_{o(k+1)} = X_{ok}'\underline{1}/n_o$$

$M_k \geq M_{k+1}$. Then

$$\underline{x}'_{k+1}M_{k+1}^{-1}\underline{x}_{k+1} \geq \underline{x}_k'M_k^{-1}\underline{x}_k$$

$\underline{x}_k$, $\underline{x}_{k+1}$ and M_k, M_{k+1} are respectively, the end points and information matrices at the kth and (k + 1)th iterations.

Proof: By the exchange rule $\underline{x}_{k+1} \leq \underline{x}_k$.

Let

$$\underline{x}_{k+1} = \underline{x}_k - \underline{u}_k$$

Then

$$\underline{x}'_{k+1}M_{k+1}^{-1}\underline{x}_{k+1} = \underline{x}_k' M_{k+1}^{-1}\underline{x}_k + 2\underline{u}_k'M_{k+1}^{-1}\underline{u}_k + \underline{u}_k' M_{k+1}^{-1}\underline{u}_k$$

On setting $\underline{c} = 2\underline{u}_kM_{k+1}^{-1}$ we have

$$\underline{x}'_{k+1}M_{k+1}^{-1}\underline{x}_{k+1} = \underline{x}_k' M_{k+1}^{-1}\underline{x}_k + 2\underline{c}'\underline{x}_k + c'M_{k+1}\underline{c}$$

$$\text{R.H.S.} \Rightarrow \underline{x}_k' M_{k+1}^{-1}\underline{x}_k \geq 2\underline{c}'\underline{x}_k - c'M_{k+1}\underline{c}$$

$$= \max\{2\underline{c}'\underline{x}_k - \underline{c}'M_{k+1}\underline{c}, \ \underline{c} \in R^n\} \geq 0, \quad \text{from 1.1.5.}$$

Therefore

$$\underline{x}'_{k+1}M_{k+1}^{-1}\underline{x}_{k+1} \geq \underline{x}_k'M_{k+1}^{-1}\underline{x}_k$$

$$\Rightarrow \underline{x}'_{k+1}M_{k+1}^{-1}\underline{x}_{k+1} \geq \underline{x}_k'M_k^{-1}\underline{x}_k$$

Theorem 2.6: First – Order Necessary Condition.

Given $\underline{x}^*$ to be a minimizer of an LP problem. Then,

$$\underline{x}^{*\prime}M_{\bullet}^{-1}\underline{x}^* = \max_k \min\{\underline{x}'M_k^{-1}\underline{x}, \ \underline{x} \in \tilde{X}\}$$

is a first – order necessary condition to be satisfied by $\underline{x}^*$; $M_{\bullet}$ is the information matrix at the point where $\underline{x}_k$ converges to $\underline{x}^*$, $\underline{x}_k$ and M_k are the end point and information matrix at the kth iteration.

Proof: Let $\lim_{k\to\infty} M_k = M_{\bullet}$

i.e $M_1 \geq M_2 \geq \ldots\ldots\geq M_{\bullet}$ and let

$$M_k = M_{\bullet} + \underline{d}_k\underline{d}_k', \ \underline{d}_k \geq 0$$

From lemma 1.2.

$$M_k^{-1} = M_{\bullet}^{-1} - \underline{v}_k\underline{v}_k', \quad v_k = n_o^{1/2}\alpha_{ok}(1 + w_k)^{1/2}M_k^{-1}\underline{d}_k, \ w_k = n_o\alpha_{ok}^2\underline{d}_k'M_{\bullet}^{-1}\underline{d}_k$$

$$\underline{x}_k' M_k^{-1} \underline{x}_k = \underline{x}_k' [M_*' - \underline{v}_k \underline{v}_k'] \underline{x}_k$$
$$= \underline{x}_k' M_*^{-1} \underline{x}_k - \underline{x}_k' \underline{v}_k \underline{v}_k' \underline{x}_k$$
$$\Rightarrow \quad \underline{x}_k' M_k^{-1} \underline{x}_k = \underline{x}_k' M_k^{-1} \underline{x}_k - \underline{x}_k' \underline{v}_k \underline{v}_k' \underline{x}_k$$
$$\Rightarrow \quad \underline{x}_k' M_*^{-1} \underline{x}_k \geq \underline{x}_k' M_k^{-1} \underline{x}_k$$

since $\underline{x}_k' \underline{v}_k \underline{v}_k' \underline{x}_k \geq 0$

$$\Rightarrow \quad \underline{x}^{*\prime} M_*^{-1} \underline{x}^* \geq \underline{x}_k' M_k^{-1} \underline{x}_k$$

since, from lemma 2.6 $\underline{x}'_{k+1} M^{-1}_{k+1} \underline{x}_{k+1} \geq \underline{x}_k' M_k^{-1\prime} \underline{x}_k$, then,

$$x^{*\prime} M_*^{-1} \underline{x}^* = \max\{\underline{x}_k' M_k^{-1} \underline{x}_k\}$$

But from Theorem 2.4

$$\underline{x}_k' M_k^{-1} \underline{x}_k = \min\{\underline{x}_k' M_k^{-1} \underline{x}\}, \quad x \in S_x$$

Therefore

$$x^{*\prime} M_*^{-1} \underline{x}^* = \max\min\{\underline{x}_k' M_k^{-1} \underline{x}, \quad \underline{x} \in \tilde{X}\}.$$

Using the fact that
min f(x) = max(-f($\underline{x}$)) we have

$$x^{*\prime} M_*^{-1} \underline{x}^* = \min_k \max\{\underline{x}_k' M_k^{-1} \underline{x}, \quad \underline{x} \in \tilde{X}\}.$$

as a first – order necessary condition to be satisfied for the existence of optimizers of LP problems. That is, the optimizer of an LP problem has both the maxmin and minmax properties, which could be reached either through minimization or maximization.

3.0 Conclusion: Illustration

The optimality conditions considered in this paper are here demonstrated with an illustration. It is here reminded that the development of the optimality conditions are based on the Linear Exchange Algorithm (LEA), a method of solving LP problems. Applying the basic steps of the LEA to the problem

$$\begin{array}{lll} \text{minimize} & f(\underline{x}) = 3x_1 + 2x_2 & \\ \text{subject to} & 2x_1 + x_1 & \geq 6 \\ & x_1 + x_2 & \geq 4 \\ & x_1 + 2x_2 & \geq 6 \\ & x_1, x_2 & \geq 0 \end{array}$$

(see Umoren, 2000), we give table 3.1 for k = 1,2,3,4 iterations.

Table 3.1: d-functions at the support points and end points for four different iterations of the Linear Exchange Algorithm

	x_1	x_2	$d_1(x,\xi_n)$			x_1	x_2	$d_2(x,\xi_n)$
	1.00	4.00	0.2625			1.00	4.00	0.3152
	3.00	1.50	0.3124			3.00	1.50	0.2752
	4.00	1.00	0.4252			2.24	1.88	0.2861
$\underline{x}_1$	2.24	1.88	0.2382		$\underline{x}_2$	1.85	2.31	0.2695
$\bar{\underline{x}}_{01}$	2.67	2.17	0.3309		$\bar{\underline{x}}_{02}$	2.08	2.46	0.3245
	x_1	x_2	$d_3(x,\xi_n)$			x_1	x_2	$d_4(x,\xi_n)$
	1.00	4.00	0.3620			2.24	1.88	0.3428
	2.24	1.88	0.3256			1.85	2.31	0.3228
	1.85	2.31	0.3694			1.65	2.70	0.3344
$\underline{x}_3$	1.65	2.70	0.3204		$\underline{x}_4$	1.87	2.27	0.3216
$\bar{\underline{x}}_{03}$	1.70	2.75	0.3361		$\bar{\underline{x}}_{04}$	1.91	2.30	0.3331

Using the results in table 3.1, we notice the following:

(i) The d-function at the end point of the kth iteration is minimum compared to the d-functions at the support points of the design matrix.

(ii) The sequence $\left\{ \underline{x}_k' M_k^{-1} \underline{x}_k \right\}_{k=1}^{\infty}$ is non-decreasing; i.e the d-function at the minimizer $\underline{x}_k = \underline{x}^*$ is the maximum of the minimum d-functions for the k different iterations.

References

Pazman, A., 1986. Foundation of optimum Experimental Design. D- Reidal publishing company, Boston, 225 pp

Raghavarao, D., 1971. Construction and Combinatorial Problems in design of Experiments. John Wiley and Sons, New York, 226pp.

Rao, C. R. , 1965. Linear Statistical Inference and its Applications. John Wiley and Sons Inc, new York, 552 pp

Umoren, M. U., 2000. Application of optimal design theory to the solution of a constrained optimization problem. Journal of Natural and Applied Sciences. 1:78- 89.