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ON FINDING THE NON-ISOMORPHISM CLASSES OF THE (nxn)/k SEMI-LATIN SQUARES

P. E. CHIGBU

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ABSTRACT

We give and prove group-theoretic lemmas which would enable one find the non-isomorphism classes of the (nxn)/k semi-Latin squares. These lemmas are most useful when the sizes and number of semi-Latin squares under consideration are large, and the computing method via the 'nauty package' for finding their isomorphism classes, given by Bailey and Chigbu, is not readily available.

Key words and phrases: *Concurrence parameters, enumeration, semi-Latin square, non-isomorphism, Trojan square.*

1. INTRODUCTION

Firstly, we define an (nxn)/k semi-Latin square as an arrangement of nk different letters in an nxn square array such that each row-column intersection contains k letters and each letter occurs once in each row and each column. For the purpose of this paper, a semi-Latin square shall be regarded as a family of nk permutations of n objects subject to some restrictions. This class of combinatorial object is very essential in statistical, agricultural and industrial experimentation and household consumer surveys, as can be found in the literature see, for example, Bailey (1992).

The first attempt to classify semi-Latin squares into isomorphism classes was made by Preece and Freeman in 1983, and this was specifically for the (4x4)/2 semi-Latin squares. Recently, Bailey and Chigbu (1997) gave a systematic method of identifying the isomorphism classes of semi-Latin squares when n and k are small. In particular, their nauty package method of enumerating the isomorphism classes of semi-Latin squares has remained invaluable for any n and k. It is possible to use the nauty package for this exercise because semi-Latin squares can be represented as graphs which are known to be amenable to this package. The nauty package could only be available via the file transfer protocol (ftp) of the world-wide web network for the countries that are hooked on.

But where the nauty package is not readily available and there exist many squares constructed via the algorithm of Chigbu (1995) and Bailey and Chigbu (1997), which need to be grouped into various isomorphism classes, it is worthwhile to first identify the squares that surely belong to different isomorphism classes before further checks for isomorphism. These further checks for isomorphism include

(a) permuting the rows, columns and symbols within row-column intersection as well as relabelling the letters/symbols of one square to obtain the other and/or

(b) a weak concept, where in addition to (a) above, the rows and columns of the squares under consideration are allowed to be interchanged, see Bailey and Chigbu (1997).

The purpose of this paper, therefore, is to highlight the procedure that surely guarantees the identification of the non-isomorphic semi-Latin squares from a given collection of such squares constructed using the above cited approach. By doing this, users could quickly distinguish between any two squares. In this regard, group-theoretic lemmas with proofs are given as well as illustrative examples.

2. PRELIMINARIES

2.1 Relevant Notations

Suppose we have an (nxn)/k *semi-Latin square* denoted as Λ. Let $Y^{\hat{}}$ be set of letters in Λ. We define $\Lambda(i,j)$ as the set of letters in $Y^{\hat{}}$ in row i and column j of Λ for $i,j \in \{1, \ldots, n\}$. Then each letter $y \in Y^{\hat{}}$ gives rise to a permutation $\pi_y^{\hat{}}$ in S_n based on $\pi_y^{\hat{}}(i) = j$ if $y \in \Lambda(i,j)$ for all $i,j \in [1,n]$; where S_n is the set of all permutations of n symbols, $\{1, \ldots, n\}$, called the *symmetric group* on n symbols of order n!. We shall call the list of all π_y in S_n arranged according to $\pi_y(i) = j$ in a square array, a *qualitative array*.

Also, each $\sigma \in S_n$ gives rise to a subset $X_\sigma^{\hat{}}$ or $Y^{\hat{}}$ by

$$X_\sigma^{\hat{}} = \left\{ y \in Y^{\hat{}} : \pi_y^{\hat{}} = \sigma \right\}.$$

We write $N_\sigma^{\hat{}}$ for $|X_\sigma^{\hat{}}|$ or N_σ for $|X_\sigma^{\hat{}}|$ as long as there is no ambiguity of usage. Then, it can easily be seen that

$$\sum_{\sigma \in S_n : \sigma(i) = j} N_\sigma^{\hat{}} = k \forall \quad i, j \in \{1, \ldots, n\}. \qquad (1)$$

P. E. CHIGBU, Department of Statistics, University of Nigeria, Nsukka, Nigeria

On the other hand, given $N_\sigma : \sigma \in S_n$ which satisfies equation (1), there exists Λ such that

$$N_\sigma^{'} = N_\sigma \quad \text{for all } \sigma \in S_n .$$

It is also possible to partition the elements of S_n into subsets such that

$$\Lambda_0 \cup \Lambda_1 \cup \ldots \cup \Lambda_k = S_n ,$$

where

$$\Lambda_r = \left\{\sigma \in S_n : N_\sigma^{'} = r\right\}.$$

Example 1: Suppose n = 4, k = 2 and we have the following array labelled with Λ in Figure 1. Then, the set of letters in Λ is $Y = \{A, \gamma, B, \alpha, C, \beta, D, \delta\}$

Λ =

	1	2	3	4
1	Aγ	Bα	Cβ	Dδ
2	Bδ	Aβ	Dα	Cγ
3	Cα	Dγ	Aδ	Bβ
4	Dβ	Cδ	Bγ	Aα

Figure 1: A (4x4)/2 semi-Latin square Λ.

$\pi_\gamma = (243)$, $\pi_C = (13)(24)$; $L_{(243)} = \{\gamma\}$, $L_{(13)(24)} = \{C\}$; $N_{(243)} = 1$, $N_{(13)(24)} = 1$.

2.2 Harmonising Notations for Example 1 with respect to S_4

In Table 1, for $\pi \in S_4$ we write $\tilde{\pi}$ for N_π ; of course, the permutations a, c, h, k, n, t, u, x are the permutations of S_4 given in Appendix 1. For the semi-Latin square in Figure 1, there exists, therefore, the following list of N_π for $\pi \in S_4$:

$\tilde{a} = 1, \tilde{b} = 0, \tilde{c} = 1, \tilde{d} = 0, \tilde{e} = 0, \tilde{f} = 0, \tilde{g} = 0, \tilde{h} = 1, \tilde{i} = 0,$
$\tilde{k} = 1, \tilde{l} = 0, \tilde{j} = 0, \tilde{k} = 1, \tilde{l} = 0, \tilde{m} = 0, \tilde{n} = 1, \tilde{o} = 0, \tilde{p} = 0, \tilde{q} = 0, \tilde{r} = 0,$
$\tilde{s} = 0, \tilde{t} = 1, \tilde{u} = 1, \tilde{v} = 0, \tilde{w} = 0, \tilde{x} = 1.$

π_y	y	$N\pi_y$	π
π_A = id	A	1	$\tilde{a}$
π_γ = (243)	γ	1	$\tilde{c}$
π_B = (12)(34)	B	1	$\tilde{h}$
π_α = (123)	α	1	$\tilde{k}$
π_β = (134)	β	1	$\tilde{n}$
π_δ = (142)	δ	1	$\tilde{t}$
π_C = (13)(24)	C	1	$\tilde{u}$
π_D = (14)(23)	D	1	$\tilde{x}$

Table 1: Table of Notations

On the other hand, given the above list of N_π for $\pi \in S_4$, there exists one letter which occurs in blocks (i, a(i)) for i = 1, ..., 4 of Figure 1. We can call this letter L_1. Also, given the same list above, there exists one letter which occurs in blocks (i, c(i)) for i = 1, ..., 4 of Figure 1. We can also call this letter L_2, etc. We can therefore say that the semi-Latin square Λ in Figure 1 is the same as the semi-Latin square, M, in Figure 2 except for the relabelling of letters.

L_1 L_2	L_3 L_4	L_5 L_6	L_7 L_8
L_3 L_8	L_1 L_6	L_7 L_4	L_5 L_2
L_5 L_4	L_7 L_2	L_1 L_8	L_3 L_6
L_7 L_6	L_5 L_8	L_3 L_2	L_1 L_4

Figure 2: The relabelled (4x4)/2 semi-Latin square, M.

In general, we call a list of values of N_σ for σ in S_n which satisfies equation (1), a *quantitative array* for (n, k) or simply a *quantitative array*.

3. METHODS THAT EMPHASIZE THE NON-ISOMORPHISM OF SEMI-LATIN SQUARES

Some methods for examining whether a set of semi-Latin square designs belongs to different isomorphism classes or not exist. As implied earlier, in some situations, these methods are not conclusive.

We shall now examine two such methods: the *combinatorial parameters'* and *variety-concurrence graphs'* methods.

Let a_i be the number of letter-pairings that occur i times (i = 0, 1, ..., n) within the row-column intersections of a semi-Latin square. The family $(a_i)_{i=1}^{n}$ of any semi-Latin square is known as the *combinatorial parameters* of that square: see Preece and Freeman (1983).

Aα	Bβ	Cλ	Dγ
Bβ	Aα	Dγ	Cλ
Cλ	Dγ	Aα	Bβ
Dγ	Cλ	Bβ	Aα

Figure 3: A (4x4)/2 semi-Latin square

Aα	Bβ	Cλ	Dγ
Bβ	Aα	Dγ	Cλ
Dγ	Cλ	Aα	Bβ
Cλ	Dγ	Bβ	Aα

Figure 4: A (4x4)/2 semi-Latin square

Aα	Bβ	Cλ	Dγ
Bβ	Aα	Dγ	Cλ
Cλ	Dγ	Aβ	Bα
Dγ	Cλ	Bα	Aβ

Figure 5: A (4x4)/2 semi-Latin square

Thus, the values of the combinatorial parameters for Figure 3 are

$$a_0 = 24,\ a_1 = a_2 = a_3 = 0,\ a_4 = 4.$$

For Figure 4, they are

$$a_0 = 24,\ a_1 = a_2 = a_3 = 0,\ a_4 = 4.$$

Those of Figure 5 are

$$a_0 = 22,\ a_1 = 0,\ a_2 = 4,\ a_3 = 0;\ a_4 = 2.$$

Based on the combinatorial parameters of the three semi-Latin squares (Figure 3, Figure 4 and Figure 5), it can easily be seen that Figure 3 and Figure 4 have the same combinatorial parameters but are surely not isomorphic to each other. This is because they arise by merely two-fold inflation of the 4x4 Latin squares from different isotopy classes, see, for example, Dénes and Keedwell (1974) and Preece and Freeman (1983). But, because both Figure 3 and Figure 4 have different combinatorial parameter values from Figure 5, they are surely different from Figure 5.

In this way, we conclude that the combinatorial parameters' method of classifying these squares is more useful in showing non-isomorphism than isomorphism. However, if the dimensions of the semi-Latin squares under consideration increase, this procedure involves a lot more work than usual to check non-isomorphism.

On the other hand, a *variety-concurrence* graph G(Λ) of a semi-Latin square Λ has letters or symbols as *vertices* and the number of edges between any two letters y_1 and y_2 equals the number of row-column intersections containing y_1 and y_2 in Λ.

When the variety-concurrence graphs of semi-Latin squares are given (see, for example, Bailey (1992)), the general notion of relating graph theory to these squares is implied. At least, it is easy to differentiate an inflated semi-Latin square from a

Trojan square (a special sort of semi-Latin square) of equivalent size just by observing their variety-concurrence graphs. A *Trojan square* arises by the superposition of a set of k mutually orthogonal Latin squares on distinct sets of n letters, such that the resulting square has nk letters, each occurring in n row-column intersections, n rows and n columns.

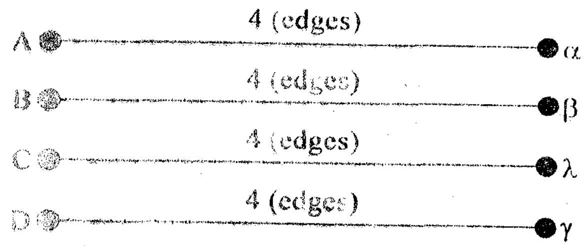

Figure 6: Variety-Concurrence Graph for Figure 3.

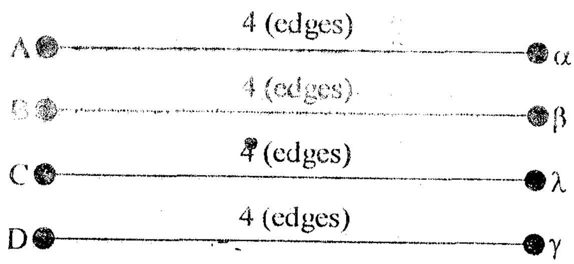

Figure 8: Variety-Concurrence Graph for Figure 5

It can, therefore, easily be seen that Figure 8 is neither isomorphic to Figure 6 nor to Figure 7. This method is also most advantageous in showing non-isomorphism classes.

4. GROUP-THEORETIC LEMMAS ON THE NON-ISOMORPHISM OF SEMI-LATIN SQUARES

Let the set Λ_r which can be partitioned into Λ_0, Λ_1, Λ_2, ..., Λ_k of various permutation cycles give rise to the semi-Latin square Λ, and also let another square M arise from M_r which can be further partitioned into M_0, M_1, M_2, ..., M_k of various permutation cycles as well. We want to check whether the semi-Latin square Λ is non-isomorphic to the square M. The concept of non-isomorphism of Λ and M implies that M cannot be obtained from Λ by permuting the rows of Λ by α, the columns by β and relabelling the letters by γ and/or interchanging its rows and columns. We define that an isomorphism between the (nxn)/k semi-Latin squares Λ and M is a triple (α, β, γ), where α, β ∈ S_n and γ is a bijection from Y^{Λ} to Y^M satisfying

$$\gamma(\Lambda(ij)) = M(\alpha(i), \beta(j)) \text{ for all } i, j \in \{1,\dots,n\}$$

called strong isomorphism by Bailey and Chigbu (1997) and/or a triple (α, β, δ), where α, β ∈ S_n and δ is a bijection from Y^{Λ} to Y^M satisfying

The variety-concurrence graph in Figure 6 represents the semi-Latin square in Figure 3. The semi-Latin square of Figure 4 is represented by the variety-concurrence graph of Figure 7 while Figure 8 is the graph of Figure 5.

As in the combinatorial parameters' method, the graphs of Figure 6 and Figure 7 are isomorphic to each other unlike the graph of Figure 8.

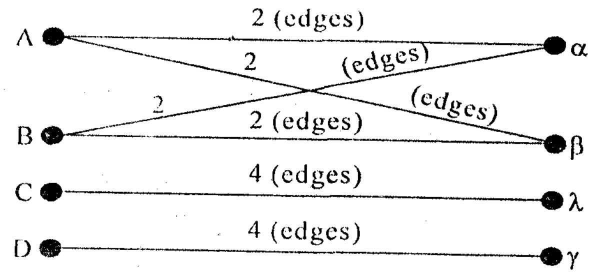

Figure 7: Variety-Concurrence Graph for Figure 4.

$$\delta(\Lambda(ij)) = M(\beta(j), \alpha(i)) \text{ for all } i, j \in \{1,\dots,n\},$$

also called weak isomorphism by Bailey and Chigbu (1997).

In the above definitions of the isomorphism of two semi-Latin squares, α and β are the row permutations and column permutations, respectively. The weak isomorphism concept also allows for the interchange of the rows and columns of the squares.

Definition: Let a set S be a subset of S_n, we define

$$S^* = \{s_z s_{z'}^{-1}; s_z \in S, s_{z'} \in S\}.$$

Now we recall the isomorphism theorem 1 of Bailey and Chigbu (1997) and state the following lemmas.

Lemma 4.1 (a) $M_r^* = \beta\Lambda_r^*\beta^{-1}$

Lemma 4.1 (b) If then Λ is non-isomorphic to M.

$$M_r^* \neq \beta\Lambda_r^*\beta^{-1}$$

Proof: (a) Suppose $\Lambda_r = \{\lambda_1, \dots, \lambda_s\} \subset S_n$ and let

$$\Lambda_r^* = \lambda_z\lambda_{z'}^{-1}; z \neq z', (1 \le z, z' \le s), \lambda_z \in \Lambda_r, \lambda_{z'} \in \Lambda_r$$

Define $\chi = \Lambda_r\alpha^{-1}$ and $M_r = \beta\chi_r$.

$$\chi_r^* = (\Lambda_r\alpha^{-1})^* = \lambda_z\alpha^{-1}(\lambda_{z'}\alpha^{-1})^{-1} = \lambda_z\alpha^{-1}\alpha\lambda_{z'}^{-1}$$
$$= \lambda_z\lambda_{z'}^{-1} = \Lambda_r^* \quad (2)$$

$$M_r^* = (\beta\chi_r)^* = (\beta\Lambda_r\alpha^{-1})^* = \beta\Lambda_r\alpha^{-1}(\beta\Lambda_r\alpha^{-1})^{-1}$$
$$= \beta\Lambda_r\alpha^{-1}\alpha\Lambda_r^{-1}\beta^{-1} = \beta\Lambda_r\alpha^{-1}(\Lambda_r\alpha^{-1})^{-1}\beta^{-1}$$
$$= \beta\chi_r^*\beta^{-1} \quad (3)$$

combining (2) and (3), $M_r^* = \beta\Lambda_r^*\beta^{-1}$.

Proof: (b)

Suppose
$$M_r{}^* \neq \beta\Lambda_r{}^*\beta^{-1},$$
$$M_r{}^* \neq \beta\Lambda_r{}^*\alpha^{-1}; \quad \beta \neq \alpha.$$

Therefore, by the above concept of non-isomorphism, there does not exist a one-to-one correspondence between Λ_r^* and M_r^*, and so there does not exist a bijection
$$\gamma: B_\Lambda \rightarrow B_M$$
such that
$$\gamma\left(B_\Lambda(i,j)\right) = B_M\left(\alpha(i), \beta(j)\right);$$
where B_Λ are the set of letters of Λ which form the corresponding bijection with the set of letters B_M of M. Hence Λ is non-isomorphic to M.

5. DEMONSTRATION AND ILLUSTRATIVE EXAMPLES

Without loss of generality, in the construction of the (4x4)/2 semi-Latin squares using the foregoing concepts and notations, suppose that J is a subset of S_4 whose elements and group action table are given in Appendices 1 and 2 and that N_σ has been specified such that
$$\sum_{\sigma \in J: \sigma(i) = j} N_\sigma \leq 2 \text{ for } i,j \in \{1,\ldots,4\}; \tag{4}$$
the need may usually arise for adjoining a subset K of $S_4 \setminus J$ (A\B means an inclusion in A which is not in B) to some Λ_r (r = 0,1,2) such that
$$\sum_{\sigma \in (J \cup K): \sigma(i) = j} N_\sigma = 2 \text{ for } i,j \in \{1,\ldots,4\}; \tag{5}$$
see, also, Bailey and Chigbu (1997). Furthermore, if a choice has to be made between K_1 and K_2, say, for K, and by our definitions of isomorphism, there exist α, β in S_4 such that $\alpha^{-1}J\beta = J$ and $\alpha^{-1}K_1\beta = K_2$ then one of K_1 and K_2 would be selected and the other left out.

If $|\Lambda_1| = 8$, in which case, we assume the identity element $a \in \Lambda_1$, we could have among others the following different sets of permutations
$$\Lambda_1 = \{a, c, k, l, n, u, t, s\}$$
$$M_1 = \{a, c, k, h, n, v, w, t\}$$
which would lead to the configurations or quantitative arrays in Figures 9 and 10 accordingly

$\Lambda_1 =$

$\tilde{a} = 1$, $\tilde{c} = 1$	$\tilde{k} = 1$, $\tilde{l} = 1$	$\tilde{n} = 1$, $\tilde{u} = 1$	$\tilde{t} = 1$, $\tilde{s} = 1$
$\tilde{t} = 1$, $\tilde{s} = 1$	$\tilde{a} = 1$, $\tilde{n} = 1$	$\tilde{k} = 1$, $\tilde{l} = 1$	$\tilde{c} = 1$, $\tilde{u} = 1$
$\tilde{k} = 1$, $\tilde{u} = 1$	$\tilde{c} = 1$, $\tilde{s} = 1$	$\tilde{a} = 1$, $\tilde{t} = 1$	$\tilde{l} = 1$, $\tilde{n} = 1$
$\tilde{l} = 1$, $\tilde{n} = 1$	$\tilde{t} = 1$, $\tilde{u} = 1$	$\tilde{c} = 1$, $\tilde{s} = 1$	$\tilde{a} = 1$, $\tilde{k} = 1$

Figure 9

$M_1 =$

$\tilde{a} = 1$, $\tilde{c} = 1$	$\tilde{k} = 1$, [illegible]	$\tilde{n} = 1$, $\tilde{v} = 1$	$\tilde{w} = 1$, $\tilde{t} = 1$
$\tilde{h} = 1$, $\tilde{t} = 1$	$\tilde{a} = 1$, $\tilde{n} = 1$	$\tilde{k} = 1$, $\tilde{w} = 1$	$\tilde{c} = 1$, $\tilde{v} = 1$
$\tilde{k} = 1$, $\tilde{w} = 1$	$\tilde{c} = 1$, $\tilde{v} = 1$	$\tilde{a} = 1$, $\tilde{t} = 1$	$\tilde{n} = 1$, $\tilde{h} = 1$
$\tilde{n} = 1$, $\tilde{v} = 1$	$\tilde{w} = 1$, $\tilde{t} = 1$	$\tilde{c} = 1$, $\tilde{h} = 1$	$\tilde{a} = 1$, $\tilde{k} = 1$

Figure 10

$$\text{For } \Lambda_1; \lambda_z\lambda_{z'}^{-1}(z \neq z') = \left\{\begin{array}{lllllll}
ac^{-1}, & ak^{-1}, & al^{-1}, & an^{-1}, & au^{-1}, & at^{-1}, & as^{-1} \\
ca^{-1}, & ck^{-1}, & cl^{-1}, & cn^{-1}, & cu^{-1}, & ct^{-1}, & cs^{-1} \\
ka^{-1}, & kc^{-1}, & kl^{-1}, & kn^{-1}, & ku^{-1}, & kt^{-1}, & ks^{-1} \\
la^{-1}, & lc^{-1}, & lk^{-1}, & ln^{-1}, & lu^{-1}, & lt^{-1}, & ls^{-1} \\
na^{-1}, & nc^{-1}, & nk^{-1}, & nl^{-1}, & nu^{-1}, & nt^{-1}, & ns^{-1} \\
ua^{-1}, & uc^{-1}, & uk^{-1}, & ul^{-1}, & un^{-1}, & ut^{-1}, & us^{-1} \\
ta^{-1}, & tc^{-1}, & tk^{-1}, & tl^{-1}, & tn^{-1}, & tu^{-1}, & ts^{-1} \\
sa^{-1}, & sc^{-1}, & sk^{-1}, & sl^{-1}, & sn^{-1}, & su^{-1}, & st^{-1}
\end{array}\right\}$$

$$= \left\{ \begin{array}{ccccccc} af, & ao, & as, & aq, & au, & aj, & al \\ ca, & co, & cs, & cq, & cu, & cj, & cl \\ ka, & kf, & ks, & kq, & ku, & kj, & kl \\ la, & lf, & lo, & lq, & lu, & lj, & ll \\ na, & nf, & no, & ns, & nu, & nj, & nl \\ ua, & uf, & uo, & us, & uq, & uj, & ul \\ ta, & tf, & to, & ts, & tq, & tu, & tl \\ sa, & sf, & so, & ss, & sq, & su, & sj \end{array} \right\}$$

due to the group action table given in Appendix 2.

Therefore

$$\Lambda_1^* = \left\{ \begin{array}{ccccccc} f & o & s & q & u & j & l \\ c & h & p & u & k & x & r \\ k & h & r & x & c & u & p \\ l & i & r & e & s & p & u \\ n & u & x & e & t & h & i \\ u & o & f & l & j & q & s \\ t & x & u & i & h & n & e \\ s & r & i & u & p & l & e \end{array} \right\}$$

and so the set $\Lambda_1^* = \{f,o,s,q,u,j,l,c,h,p,k,x,r,i,e,n,t\}$, is made up of two 2-cycle permutations (r,e); eight 3-cycle permutations (f,o,q,j,c,k,n,t); four 4-cycle permutations (s,l,p,i), and three 2^2-cycle permutations (u,h,x).

Similarly, M_1; $m_z m_{z'}^{-1}$ ($z \neq z'$) would be deduced such that we obtain the set $M_1^* = \{f,o,h,q,w, v,j,c,t,u,m,l,x,k,n,s,d\}$ made up of two 2-cycle permutations (m,d); eight 3-cycle permutations (f,o,q, j,c,t,k,n); four 4-cycle permutations (w,v,l,s), and three 2^2-cycle permutations (h, u, x).

Considering the above illustration for the two configurations, surely there exist some $\alpha, \beta \in S_4$ such that Λ_1^* is conjugated to M_1^* since Λ_1^* and M_1^* are made up of the same numbers of equivalent-cycle permutations. So the semi-Latin squares Λ and M arising from the use of Λ_1 and M_1, respectively belong to the same (strong) isomorphism class otherwise they are non-isomorphic. The case for weak isomorphism classes of two semi-Latin squares could easily be deduced by recognising that the effect of transposing Λ to get M is possible by replacing each element of Λ_1 by its inverse. Hence weak non-isomorphism classes can easily be deduced.

6. CONCLUSION

The above procedures of detecting non-isomorphism would need to be applied collectively or otherwise to reduce the cumbersomeness of enumerating semi-Latin squares of a given size in the first instance. Subsequently, the conclusive procedure of permuting rows, columns and letters/symbols within row-column intersection, relabelling symbols and interchanging of rows and columns of one square to obtain the other would easily be applied to pin down non-isomorphism classes in situations where the automated enumeration procedure via nauty is not possible.

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APPENDIX I

PERMUTATIONS OF S_4

PERMUTATIONS OF S_4

$$a=\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}=id;\quad b=\begin{pmatrix}1&2&3&4\\1&2&4&3\end{pmatrix}=(34);\quad c=\begin{pmatrix}1&2&3&4\\1&4&2&3\end{pmatrix}=(243)$$

$$d=\begin{pmatrix}1&2&3&4\\1&4&3&2\end{pmatrix}=(24);\quad e=\begin{pmatrix}1&2&3&4\\1&3&2&4\end{pmatrix}=(23);\quad f=\begin{pmatrix}1&2&3&4\\1&3&4&2\end{pmatrix}=(234)$$

$$g=\begin{pmatrix}1&2&3&4\\2&1&3&4\end{pmatrix}=(1,2);\quad h=\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}=(12)(34);\quad i=\begin{pmatrix}1&2&3&4\\2&4&1&3\end{pmatrix}=(1243)$$

$$j=\begin{pmatrix}1&2&3&4\\2&4&3&1\end{pmatrix}=(124);\quad k=\begin{pmatrix}1&2&3&4\\2&3&1&4\end{pmatrix}=(123);\quad l=\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}=(1234)$$

$$m=\begin{pmatrix}1&2&3&4\\3&2&1&4\end{pmatrix}=(13);\quad n=\begin{pmatrix}1&2&3&4\\3&2&4&1\end{pmatrix}=(134);\quad o=\begin{pmatrix}1&2&3&4\\3&1&2&4\end{pmatrix}=(132)$$

$$p=\begin{pmatrix}1&2&3&4\\3&1&4&2\end{pmatrix}=(1342);\quad q=\begin{pmatrix}1&2&3&4\\4&2&1&3\end{pmatrix}=(143);\quad r=\begin{pmatrix}1&2&3&4\\4&2&3&1\end{pmatrix}=(14)$$

$$s=\begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}=(1432);\quad t=\begin{pmatrix}1&2&3&4\\4&1&3&2\end{pmatrix}=(142);\quad u=\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}=(13)(24)$$

$$v=\begin{pmatrix}1&2&3&4\\3&4&2&1\end{pmatrix}=(1324);\quad w=\begin{pmatrix}1&2&3&4\\4&3&1&2\end{pmatrix}=(1423);\quad x=\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}=(14)(23)$$

Above, the 24 permutations of S_4 are labelled arbitrarily from the identity, a, to x. Apart from the permutation 'a' which has a two-row representation and just the notation 'id', other permutations are given using the two-row representation and their equivalent cycle representation.

The table of notations, Table 1, for Figure 1 can be arranged side by side in the following (4x4) arrays.

$\tilde{a}\tilde{c}$	$\tilde{k}\tilde{h}$	$\tilde{n}\tilde{u}$	$\tilde{x}\tilde{t}$
$\tilde{t}\tilde{h}$	$\tilde{a}\tilde{n}$	$\tilde{k}\tilde{x}$	$\tilde{c}\tilde{u}$
$\tilde{k}\tilde{u}$	$\tilde{c}\tilde{x}$	$\tilde{a}\tilde{t}$	$\tilde{n}\tilde{h}$
$\tilde{x}\tilde{n}$	$\tilde{t}\tilde{u}$	$\tilde{c}\tilde{h}$	$\tilde{a}\tilde{k}$

1 2	3 4	5 6	7 8
4 8	1 5	3 7	2 6
3 6	2 7	1 8	4 5
5 7	6 8	2 4	1 3

based on the following general (4x4) arrays (Figures 11 and 12) which lie side by side and which we have respectively called the Qualitative Array and the Quantitative Array.

abc def	ghi jkl	mno puv	qrs twx
gho pst	abm nqr	efk lwx	cdi juv
ikm quw	ceo svx	adg jrt	bfh lnp
jln rvx	dfp tuw	bch iqs	aeg kmo

Figure 11: The Qualitative Array

$\tilde{a}\tilde{b}\tilde{c}$ $\tilde{d}\tilde{e}\tilde{f}$	$\tilde{g}\tilde{h}\tilde{i}$ $\tilde{j}\tilde{k}\tilde{l}$	$\tilde{m}\tilde{n}\tilde{o}$ $\tilde{p}\tilde{u}\tilde{v}$	$\tilde{q}\tilde{r}\tilde{s}$ $\tilde{t}\tilde{w}\tilde{x}$
$\tilde{g}\tilde{h}\tilde{o}$ $\tilde{p}\tilde{s}\tilde{t}$	$\tilde{a}\tilde{b}\tilde{m}$ $\tilde{n}\tilde{q}\tilde{r}$	$\tilde{e}\tilde{f}\tilde{k}$ $\tilde{l}\tilde{w}\tilde{x}$	$\tilde{c}\tilde{d}\tilde{i}$ $\tilde{j}\tilde{u}\tilde{v}$
$\tilde{i}\tilde{k}\tilde{m}$ $\tilde{q}\tilde{u}\tilde{w}$	$\tilde{c}\tilde{e}\tilde{o}$ $\tilde{s}\tilde{v}\tilde{x}$	$\tilde{a}\tilde{d}\tilde{g}$ $\tilde{j}\tilde{r}\tilde{t}$	$\tilde{b}\tilde{f}\tilde{h}$ $\tilde{l}\tilde{n}\tilde{p}$
$\tilde{j}\tilde{l}\tilde{n}$ $\tilde{r}\tilde{v}\tilde{x}$	$\tilde{d}\tilde{f}\tilde{p}$ $\tilde{t}\tilde{u}\tilde{w}$	$\tilde{b}\tilde{c}\tilde{h}$ $\tilde{i}\tilde{q}\tilde{s}$	$\tilde{a}\tilde{e}\tilde{g}$ $\tilde{k}\tilde{m}\tilde{o}$

Figure 12: The Quantitative Array

The permutations involved are arranged according to the following criteria:

> By using the two-row form to represent the permutations of S_4, the first row of any permutation (in its natural order) represents the rows while its second row represents the columns of the array in which the particular permutation falls for a given row.

In other words, the second row of any permutation (though, not always in the natural order but actually depends on the particular permutation's bijection function) represents the columns which the permutations of S_4 occupy (called permutation positions) in the general qualitative array.

APPENDIX 2

GROUP ACTION TABLE OF S_4

o	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x
a	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x
b	b	a	e	f	c	d	h	g	k	l	i	j	q	r	s	t	m	n	o	p	w	x	u	v
c	c	d	f	e	b	a	s	t	w	x	q	r	i	j	h	g	u	v	p	o	k	l	m	n
d	d	c	b	a	f	e	t	s	q	r	w	x	u	v	p	o	i	j	h	g	m	n	k	l
e	e	f	d	c	a	b	o	p	u	v	m	n	k	l	g	h	w	x	t	s	i	j	q	r
f	f	e	a	b	d	c	p	o	m	n	u	v	w	x	t	s	k	l	g	h	q	r	i	j
g	g	h	i	j	k	l	a	b	c	d	e	f	o	p	m	n	s	t	q	r	v	u	x	w
h	h	g	k	l	i	j	b	a	e	f	c	d	s	t	q	r	o	p	m	n	x	w	v	u
i	i	j	l	k	h	g	q	r	x	w	s	t	c	d	b	a	v	u	n	m	e	f	o	p
j	j	i	h	g	l	k	r	q	s	t	x	w	v	u	n	m	c	d	b	a	o	p	e	f
k	k	l	j	i	g	h	m	n	v	u	o	p	e	f	a	b	x	w	r	q	c	d	s	t
l	l	k	g	h	j	i	n	m	o	p	v	u	x	w	r	q	e	f	a	b	s	t	c	d
m	m	n	v	u	o	p	k	l	j	i	g	h	a	b	e	f	r	q	x	w	d	c	t	s
n	n	m	o	p	v	u	l	k	g	h	j	i	r	q	x	w	a	b	e	f	t	s	d	c
o	o	p	u	v	m	n	e	f	d	c	a	b	g	h	k	l	t	s	w	x	j	i	r	q
p	p	o	m	n	u	v	f	e	a	b	d	c	t	s	w	x	g	h	k	l	r	q	j	i
q	q	r	x	w	s	t	i	j	l	k	h	g	b	a	c	d	n	m	v	u	f	e	p	o
r	r	q	s	t	x	w	j	i	h	g	l	k	n	m	v	u	b	a	c	d	p	o	f	e
s	s	t	w	x	q	r	c	d	f	e	b	a	h	g	i	j	p	o	u	v	l	k	n	m
t	t	s	q	r	w	x	d	c	b	a	f	e	p	o	u	v	h	g	i	j	n	m	l	k
u	u	v	n	m	p	o	w	x	r	q	t	s	d	c	f	e	j	i	l	k	a	b	g	h
v	v	u	p	o	n	m	x	w	t	s	r	q	j	i	l	k	d	c	f	e	g	h	a	b
w	w	x	r	q	t	s	u	v	n	m	p	o	f	e	d	c	l	k	j	i	b	a	h	g
x	x	w	t	s	r	q	v	u	p	o	n	m	l	k	j	i	f	e	d	c	h	g	b	a