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THE COMPUTATION OF LINEAR TRIANGULAR SYSTEM MATRICES IN THE FINITE ELEMENT METHOD

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ABSTRACT

An algorithm is developed for generating the system matrices for the Finite Element Method of solving some classes of second order partial differential equations problems using linear triangular elements. This algorithm reduces the complexity normally associated with the finite element approximation and makes the powerful technique accessible to more people even on personal computers.

KEY WORDS: Triangular elements, system matrices, Finite Element Method.

INTRODUCTION

The Finite Element Method (FEM) for solving partial differential equation problems essentially involves (i) the division of the domain into subdomains (elements), (ii) the representation of the dependent variable element by element (nodal) parameters, and (iii) the solution of a linear system of equations in the element parameters, (Olayi, 1977).

The finite elements have two distinguishing features: the shape and functions defined on them. Thus, a linear triangular element is one whose shape is a triangle and the function is a linear piecewise polynomial. The function defined on the elements are made up of element basis functions, f_j^e :

$$f_j^e = \begin{cases} 1 \text{ at node j of element e} \\ 0 \text{ at other nodes of element e.} \end{cases}$$

and is non zero only on element e. The basis function, ϕ_j, of the finite element trial space is a piecewise function of the element basis functions of elements having j as a node. ϕ_j is therefore non zero only on elements having j as one of their nodes.

The system of equations in element (nodal) parameters is often referred to as Finite Element Equation or the System Matrix Equation, (Strang & Fix, 1973; Norrie & Vries, 1973; Olayi, 1986); and is usually of the form KP = F, KP = MP or MP + KP = F depending on whether the given problem is equilibrium, eigenvalue or propagation (Strang & Fix, 1973; Olayi 1977; Norrie & de Vries, 1973). M is called the mass matrix, while K is stiffness matrix. M and K are together referred to as system matrices. P is the vector of nodal Parameters.

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To solve the finite element equation, we need to know M and K. The usual approach is to construct the element and trial space basic functions and then find the contribution from each element. This involves a lot of book-keeping, manipulations and calculations. In this paper, we provide an algorithm which can be used to generate M and K for cases where

$$M = (m_{ij}) = \left(\iint_R \phi_i \phi_j dxdy \right) \text{ and}$$

$$K = (k_{ij}) = \left(\iint_R \left[\phi_{i,x}\phi_{j,x} + \phi_{i,y}\phi_{j,y} \right] dxdy \right)$$

ϕ_ℓ are the linear basis functions for triangular elements on R and $\phi_{\ell,t}$ denotes partial derivatives of ϕ_ℓ with respect to t. The system matrices for the finite element equation of the propagation problem whose governing equation is of the form $u_t = u_{xx} + u_{yy}$ can be generated in this way.

THE ALGORITHM

Without loss of generality, we consider the domain R to be a square domain and the finite elements as shown in Fig. 1. For an eight-element linear triangularization of a unit square (see Fig. 2.) the element basis functions are:

$f_{11} = 1 - 2x$, $f_{12} = 2x - 2y$, $f_{13} = 2y$ on e1

$f_{21} = 2 - 2x$, $f_{22} = -1 + 2x - 2y$, $f_{23} = 2y$ on e2

$f_{31} = 1 - 2y$, $f_{32} = -1 + 2x$, $f_{33} = 1 - 2x + 2y$ on e3

$f_{41} = 1 - 2y$, $f_{42} = 2x$, $f_{43} = 2y - 2x$ on e4

$f_{51} = 1 - 2x$, $f_{52} = 1 + 2x - 2y$, $f_{53} = -1 + 2y$ on e5

$f_{61} = 2 - 2x$, $f_{62} = 2x - 2y$, $f_{63} = -1 + 2y$ on e6

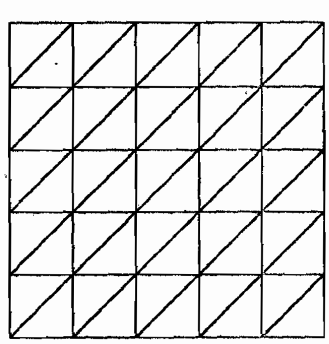

Fig. 1
LINEAR TRIANGULARIZATION ON R.

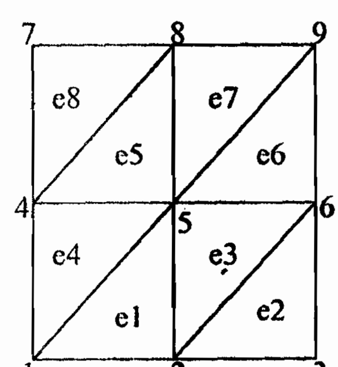

Fig. 2
AN EIGHT-ELEMENT LINEAR TRIANGULARIZATION ON R.

$f_{71} = 2 - 2y$, $f_{72} = 2x - 1$, $f_{73} = -2x + 2y$ on e7

$f_{81} = 2 - 2y$, $f_{82} = 2x$, $f_{83} = -1 - 2x + 2y$ on e8;

while the trial space basic functions are:

$$\phi_1 = \begin{cases} f_{11} \text{ on e1} \\ f_{41} \text{ on e4} \\ 0 \quad \text{elsewhere} \end{cases}, \phi_2 = \begin{cases} f_{12} \text{ on e1} \\ f_{21} \text{ on e2} \\ f_{31} \text{ on e3} \\ 0 \quad \text{elsewhere} \end{cases}$$

$$\phi_3 = \begin{cases} f_{22} \text{ on e2} \\ 0 \quad \text{elsewhere} \end{cases}, \phi_7 = \begin{cases} f_{83} \text{ on e8} \\ 0 \quad \text{elsewhere} \end{cases}$$

$$\phi_4 = \begin{cases} f_{43} \text{ on e4} \\ f_{51} \text{ on e5} \\ f_{81} \text{ on e8} \\ 0 \quad \text{elsewhere} \end{cases}, \phi_6 = \begin{cases} f_{32} \text{ on e3} \\ f_{23} \text{ on e2} \\ f_{62} \text{ on e6} \\ 0 \quad \text{elsewhere} \end{cases}$$

$$\phi_8 = \begin{cases} f_{53} \text{ on e5} \\ f_{73} \text{ on e7} \\ f_{82} \text{ on e8} \\ 0 \quad \text{elsewhere} \end{cases}, \phi_9 = \begin{cases} f_{63} \text{ on e6} \\ f_{72} \text{ on e7} \\ 0 \quad \text{elsewhere} \end{cases}$$

$$\phi_5 = \begin{cases} f_{13} \text{ on e1} \\ f_{33} \text{ on e3} \\ f_{42} \text{ on e4} \\ f_{52} \text{ on e5} \\ f_{61} \text{ on e6} \\ f_{71} \text{ on e7} \\ 0 \quad \text{elsewhere} \end{cases}$$

THEOREM

If $\phi_j(x,y), j = 1,2,\ldots,n^2$ are the basis functions of the trial space of linear triangular elements on R, then

$$\left\{ \iint_R \phi_i \phi_j dxdy \middle| i,j = 1,2,\ldots,n^2 \right\} = \{-a,-2a,0,a,2a,8a,10a,20a\}$$

$$\left\{\iint_R [\phi_{i,x}\phi_{j,x} + \phi_{i,y}\phi_{j,y}]dxdy \middle| i,j = 1,2,\ldots,n^2\right\} = \{-1,-2,0,2,4,8\}$$

where $a = \frac{h^2}{12}$, h being the size of the squares from which the triangularization is derived.

Proof (The Algorithm).

The values of $m_{ij} = \iint_R \phi_i\phi_j dxdy$ and $k_{ij} = \iint_R [\phi_{i,x}\phi_{j,x} + \phi_{i,y}\phi_{j,y}]dxdy$ depend on the position of the nodes in the triangularization, see Figs. 1 and 2. These can be grouped into eight categories as follows:

Diagonal - entries, i = j

D-1: If j is a corner node of R joined by a diagonal edge to an interior node then $m_{jj} = 8a, k_{jj} = 2$.

D-2: If j is a corner node of R other than D-1, $m_{jj} = 2a, k_{jj} = 2$.

D-3: If j is a boundary node but not a corner node, $m_{jj} = 10a, k_{jj} = 4$.

D-4: If j is not a boundary node then $m_{jj} = 20a, k_{jj} = 8$.

Off-diagonal entries, i ≠ j

0-1: If i and j are adjacent boundary node but not a corner node, $m_{ij} = -a, k_{ij} = -1$.

0-2: If i and j are adjacent diagonal nodes (nodes on an edge dividing a square into triangles), $m_{ij} = a, k_{ij} = 0$.

0-3: If i and j are adjacent nodes other than 0-1 and 0-2, then $m_{ij} = -2a, k_{ij} = -2$.

0-4: If i and j are diagonal nodes of a square but not connected by an edge or i and j are at least 2h apart, then $m_{ij} = 0 = k_{ij}$.

The above assertions can be easily established once the basis functions, ϕ_j, of the trial space are constructed and this completely determines the entries of the system matrices.

CONCLUSION

The system matrices for linear triangular elements can be generated using the steps listed under the algorithm above. All entries of the mass matrix, regardless of the size, are made up of only eight distinct numbers which depend on the size, h, of the triangularization while those of the stiffness matrix are made up of six distinct numbers, independent of the size of the triangularization. One added advantage of this is the fact that the stiffness matrix of any size can be generated from the smallest stiffness matrix involving all the entries listed under the algc ithm.

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