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VISCOUS COSMOLOGICAL MODELS WITH A VARIABLE COSMOLOGICAL TERM

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ABSTRACT

Einstein's field equations for a Friedmann-Lamaitre Robertson-Walker universe filled with a dissipative fluid with a variable cosmological term Λ described by full Israel-Stewart theory are considered. General solutions to the field equations for the flat case have been obtained. The solution corresponds to the dust free model where $p = 0$. It also admits the exponential solution for hubble parameter H which represent the period of hyperinflation or superinflation where the energy density and the cosmological term grow enormously. The temperature is regarded as a constant.

INTRODUCTION

The role played by viscosity and the consequent dissipative mechanism in cosmology has been discussed by many authors (Schweizer, 1988; Pavon *et al, 1991;* Trigner and Pavon, 1995). The expansion rate of the universe is slowed down by the scalar field from exponential to the polynomial so that there is enough time for the universe to complete the phase transition from the inflationary to the radiation dominated phase. Dissipative effect such as viscosity are of enormous important in the early stage of the evolution of the universe, particularly before the time of the nucleosynthesis (Gron, 1990).

A number of observations e.g. type IA supernovae, now compellingly suggested that the universe possesses a non zero cosmological term (Krauss and Turner, 1995). The cosmological term represents energy which is combined with the matter of the universe. For standard inflation, a universe with a cosmological term would expand faster with time because of the push from the cosmological term (Croswell, 1994). The cosmological term could be a function of time in the spatially homogenous expanding universe.

Any model of the universe should yield a lifetime greater than that of the oldest objects in it, so it is difficult for the Friedmann-Lamaitre Robertson-Walker (FLRW) models, without the cosmological term, to have an age of the universe greater than that of the oldest stars (Bagla *et al*, 1996; Davies, 1996 and Fukuyama *et al*, 1997).

The first solution of the cosmological models with time dependent G and Λ were obtained by Bertolami (1986) which were extended by Abdussatar and Vishwakama (1997). Also Arbab (1997) found several solutions similar to the ones obtained by Berman (1991) and Kalligas *et al* (1992), claiming that energy is conserved. The solution was modified by Singh *et al* (1998), where they obtained the energy density ρ as a decreasing function of time, with energy conservation.

Most of these investigations are based on Eckart theory (Hiscock and Solmonson, 1991; Beesham, 1993). However, it is known that this theory is non causal and all of its equilibrium states are not stable. Together with Prof. A. Beesham and Dr S.G. Ghosh, we have studied some solutions with variable Λ in Eckart theory and in truncated theory, which has been submitted for publication in *General Relativity and Gravitation* journal.

In this work we examine the full causal theory which solves the previously mentioned problems. A general solution of an FLRW model is obtained and discussed, where $k = 0$ and the cosmological term is a function of the hubble parameter given by $\Lambda = 3\beta H^2$.

Also the expressions for the parameters Λ, ρ, τ, ξ and Π as function of time are obtained and discussed.

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a. Theoretical consideration and calculations

The time element of spatially flat Friedmann-Lamaitre Robertson-Walker model takes the form

$$ds^2 = -dt^2 + R^2(t)(dx^2 + dy^2 + dz^2) \quad (1)$$

The energy momentum tensor for a fluid with bulk viscosity is

$$T_{ab} = \rho u_a u_b + p_{eff} h_{ab} - \Lambda g_{ab} \quad (2)$$

where ρ is the energy density, u_a is the four velocity, p_{eff} is the effective non-equilibrium pressure, Λ is the cosmological term and h_{ab} is the projection tensor defined by

$$h_{ab} = g_{ab} + u_a u_b$$

and we are using units such that $8\pi G = c = 1$.

The Einstein's field equations wuth cosmological term derived from

$$G_{ab} = T_{ab}$$

are

$$\rho = 3H^2 - \Lambda \quad (3)$$

and

$$\dot{H} = -H^2 - \frac{1}{6}\rho - \frac{1}{2}(p + \Pi) + \frac{\Lambda}{3} \quad (4)$$

where

$$H = \frac{\dot{R}}{R}$$

is the hubble parameter and the dot denotes the derivative with respect to time. The conservation equation of momentum yields the non independent equation

$$\dot{\rho} = -3(\rho + p\Pi)H \quad (5)$$

where Π is the bulk viscous pressure. We assume a linear barotropic equation of state as defined by Cavao *et al* (1992)

$$p = (\gamma - 1)\rho \quad (6)$$

where γ ia a constant within the range $1 \leq \gamma \leq \frac{4}{3}$.

b. Equation of state

Equations of state of some thermodynamical variables can be derived from kinetic theory (Stewart, 1971; Caderni and Fabri, 1977). To obtained detailed equations of state, we compare one component fluid in general relativity and in Newtonian theory as indicated by Ellis and Madsen (1991). There are some equations of state in the form of $\Sigma = \Sigma(p, v)$ where Σ is the specific internal energy density of the fluid. Mason and Kgathi (1991) have investigated the dependence of *n* and specific entropy *S* for a non-dissipative relativistic gas in collision-dominated equilibrium, where $S = S(\rho, n)$ and all the variables are defined by using the first law of thermodynamics.

$$\dot{S} = \frac{1}{nT}\dot{\rho} - \frac{\rho + p}{n^2 T} \tag{7}$$

It should be noted that S is not constant along the fluid particle world line, and not throughout the fluid in general. If S is the same constant on each world line i.e. $\dot{S} = 0$ and $S_{;a} = 0$, then the fluid is called isentropic. The relationship between the thermodynamic scalar n, ρ, p, s and T is given by Gibb's equation

$$Tds = d\left(\frac{\rho}{n}\right) + pd\left(\frac{1}{n}\right) \tag{8}$$

The equation of state depends on particular physical properties of a fluid which are deduced through microscopic physics (kinetic theory). The Gibb's equation shows that in general two thermodynamic scalars are needed as independent variables. The Gibb's equation integrability condition

$$S,T_n = S,nT$$

becomes

$$n\frac{\partial T}{\partial n} + (p+\rho)\frac{\partial T}{\partial \rho} = T\frac{\partial p}{\partial \rho}$$

By taking $T = T(n, \rho)$ and following the conservation law of energy and number density, it results in the equation of state for pressure and temperature

$$\frac{\dot{T}}{T} = -3H\frac{\partial p}{\partial \rho} \tag{9}$$

c. Causal Bulk viscosity in cosmology

The energy entropy four-flux is given by

$$S^a = sN^a - \frac{\tau \Pi^2 u^a}{2\xi T} \tag{10}$$

where s is a specific entropy, $\tau \geq 0$ is the relaxation time for bulk viscosity stress, $\xi \geq 0$ is the coefficient of bulk viscosity, $T \geq 0$ is the temperature and N^a is the number of flux given by equation

$$N^a = nu^a$$

The divergence of the four flux yields

$$S^a{}_{;a} = s_{;a}N^a + sN^a{}_{;a} - \frac{\tau \Pi \Pi_{;a} u^a}{\xi T} - \frac{\Pi^2}{2}\left(\frac{\tau u^a}{\xi T}\right)_{;a} \tag{11}$$

From equations (8), (10) and (11) we obtain the causal evolution equation for bulk viscosity

$$\Pi + \tau\dot{\Pi} = -3\xi H - \frac{\varepsilon \tau \Pi}{2}\left(3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T}\right) \tag{12}$$

For $\tau = 0$ represents the standard Eckart theory equation of relativistic irreversible thermodynamics. This theory has causality violation and pathology of unstable equilibrium states. At $\varepsilon = 0$ we have extended Israel-Stewart equation (truncated theory equation). The difference between the Eckart and extended

Israel-Stewart theory equations is that the first is a simple algebraic equation and the latter is a differential evolution equation. The non-truncated Israel-Stewart transport equation is formed when $\varepsilon = 1$. This eqution is totally different from the first two equations because it accommodates the influence of the coefficient of bulk viscosity (ξ) and temperature (T) in a fluid.

Most authors use these ad hoc equations to define the relaxation time and the coefficient of bulk viscosity:

$$\xi = \alpha\rho^{m}$$

and

$$\tau = \frac{\xi}{\rho}$$

By considering the same definition in full theory we obtain the complex value of the coefficient of bulk viscosity. According to Coley *et al* (1997), the relaxation time and the coefficient of bulk viscosity are considered as proportional to power ρ. By defining the density parameter as

$$z = \frac{\rho}{3H^2} \tag{13}$$

Then the equations for τ and ξ in terms of the density parameter are

$$\tau^{-1} = bHz^{p} \tag{14}$$

and

$$\frac{\xi}{H} = 3az^{m} \tag{15}$$

Following Arbab (1997); (1998) and Singh *et al* (1998), the variation of the cosmological term (Λ) as the function of Hubble parameter H takes the form of

$$\Lambda = 3\beta H^2 \tag{16}$$

where $\beta \neq 0$ is a constant. From equation (3), (4), (6) and (16), the bulk viscosity pressure is given by

$$\Pi = 2\dot{\Pi} - 3(\gamma - \beta)H^2 \tag{17}$$

Equations (3), (4), (5), (6), (9), (12), (13), (14), (15) and (16) result in the evolution equation for H, where $\varepsilon = 1$

$$\begin{aligned}
&\left[6\gamma(1-\beta)+3+3m(1+\gamma)+\frac{3}{2}\gamma(1-\beta)\left(\frac{1}{b^2}+1-2m\right)+2b(1-\beta)^{p}\right]\dot{H}H \\
&+9(\gamma-1)(1-\beta)\left[\frac{2-\beta}{b^2}-m(\gamma+1)\right]\dot{H}H^3-\frac{54}{b^2}\gamma[(\gamma-1)(1-\beta)]^2\dot{H}H \\
&-\left(\frac{1}{b^2}+1-2m\right)\frac{\dot{H}^2}{H}+6(\gamma-1)\left(\frac{1-\beta}{b^2}-m\right)\dot{H}^2H-\frac{18}{b^2}(\gamma-1)^2(1-\beta)H^3\dot{H}^2 \\
&+2\ddot{H}+\left[\frac{9}{2}\gamma(1-\beta)(m+\gamma)+3b\gamma(1-\beta)^{p+1}-9ab(1-\beta)^{p+m}\right]H^3 \\
&+\frac{27}{2}\gamma(1-\beta)^2(\gamma-1)\left(\frac{1}{b^2}-m\gamma\right)H^5-\frac{81}{2b^2}\gamma(1-\beta)^3(\gamma-1)H^7=0
\end{aligned} \tag{18}$$

According to Maartens (1995), the equation in the form of (18) is consistent with exponential inflation with $\Lambda = 0, \beta \neq 1$ and $H = const$. The above equation gives the inflationary solution if $\beta = 1$. If $m \neq 1$ we obtain $\rho = 0, \varepsilon = 0$ and $\tau = 0$ which corresponds to perfect fluid solution.

d. General cosmological solution

In full causal theory, equation (18) admits the exponential solution for hubble factor H in the form of

$$H = H_0 e^{\alpha t} \tag{19}$$

which leads to

$$\dot{H} = H_0 \alpha e^{\alpha t} \text{ and } \ddot{H} = \alpha^2 H_0 e^{\alpha t} \tag{20}$$

where H_0 is the constant. This solution corresponds to the hyperinflation period which was obtained by Barrow (1988). It was also obtaine by Arbab (1997) in the Bianchi I model with variable cosmological term (Λ) and G, where the density parameter remains constant. By using equations (18), (19) and (20) we obtain

$$\begin{aligned}
&[6\gamma(1-\beta)+3+3m(\gamma+1)]\alpha e^{2\alpha t} + \left[\frac{3}{2}\gamma(1-\beta)\left(\frac{1}{b^2}+1-2m\right)+2b(1-\beta)^p\right]\alpha e^{2\alpha t} \\
&+9(\gamma-1)(1-\beta)\left[\frac{2}{b^2}-\frac{\beta}{b^2}-m(\gamma+1)\right]\alpha e^{4\alpha t} \\
&+\frac{54}{b^2}\gamma(\gamma-1)^2(1-\beta)^2\alpha e^{5\alpha t} + \left[6(\gamma-1)\left(\frac{1}{b^2}-\frac{\beta}{b^2}-m\right)\alpha^2\right]\alpha e^{3\alpha t} \\
&+\left[\frac{9}{2}\gamma(1-\beta)(m+\gamma)+3b\gamma(1-\beta)^{p+1}-9ab(1-\beta)^{p+m}\right]\alpha e^{3\alpha t} \\
&-\left[\frac{18}{b^2}(\gamma-1)^2(1-\beta)\alpha^2-\frac{27}{2}\gamma(\gamma-1)(1-\beta)^2\left(\frac{1}{b^2}-m\gamma\right)\right]e^{5\alpha t} \\
&+\left[2-\left(\frac{1}{b^2}+1-2m\right)\right]e^{\alpha t}+\frac{81}{2b^2}\gamma(1-\beta)^3(\gamma-1)e^{7\alpha t}=0
\end{aligned} \tag{21}$$

From equation (6) and equation (21) we obtain seven independent equations with 7 unknowns $(\alpha, \beta, \gamma, a, b, m \text{ and } p)$

$$p = (\gamma \quad)\rho$$

$$2-\left(\frac{1}{b^2}+1-2m\right)=0$$

$$6\gamma(1-\beta)+3+3m(\gamma+1)+\frac{3}{2}\gamma(1-\beta)\left(\frac{1}{b^2}+1-2m\right)+2b(1-\beta)^p=0$$

$$9(\gamma-1)(1-\beta)\left[\frac{2}{b^2}-\frac{\beta}{b^2}-m(\gamma+1)\right]=0$$

$$\frac{54}{b^2}\gamma(\gamma-1)^2(1-\beta)^2\alpha-\frac{18}{b^2}(\gamma-1)^2(1-\beta)\alpha^2+\frac{27}{2}\gamma(\gamma-1)(1-\beta)^2\left(\frac{1}{b^2}-n\gamma\right)=0$$

$$6(\gamma-1)\left(\frac{1}{b^2}-\frac{\beta}{b^2}-m\right)\alpha^2+\left[\frac{9}{2}\gamma(1-\beta)(m+\gamma)+3b\gamma(1-\beta)^{p+1}-9ab(1-\beta)^{p+m}\right]=0$$

$$\frac{81}{2b^2}\gamma(1-\beta)^3(\gamma-1)e^{7\alpha t}=0$$

By solving these equations we obtain

$$\alpha \neq 0$$
$$m = 2x+1$$
$$b = \frac{1}{\sqrt{2m+1}}$$
$$\beta = \frac{14+6m}{9}$$
$$p = 0$$
$$\gamma = 1$$
$$a = \frac{(2b+3m+3)(1-\beta)^{1-m}}{6b}$$

where $x \in R$

Since $p = 0$ the solution represents a pressure free model (dust) which implies that

$$\rho = \rho_0 e^{\alpha_i t}$$

By using equations (3), (4), (15), (16) and (17), we can express the cosmological term (Λ), the density parameter (ρ), the bulk viscous pressure (Π), the coefficient of the bulk viscosity (ξ) and the relaxation time (τ) as the function of time.

$$\Lambda = 3\beta H_0^{\,2} e^{2\alpha t}$$
$$\rho = 3(1-\beta)H_0^{\,2}e^{2\alpha t}$$
$$\Pi = 2 - H_0\alpha e^{\alpha t} - 3(1-\beta)H_0^{\,2}e^{2\alpha t}$$
$$\xi = 3a(1-\beta)^m H_0 e^{\alpha t}$$
$$\tau = \frac{1}{bH_0}e^{-\alpha t}$$

The cosmological term, the coefficient of the bulk viscosity stress and the density parameter increases with time. This indicates that the age of the universe can be deduced from these three components. In this model we can regard the value of Λ as the measure of the age of the stars.

DISCUSSION AND CONCLUSION

In this work we solved the Einstein's equations for the viscous flat FRLW universe with variable cosmological term by using full causal theory. The exponential solution for the hubble parameters are obtained which correspond to hyperinflation periods. During this epoch, the density parameter, the cosmological term and the coefficient of the bulk viscosity grow very enormously with time, while the temperature is constant. The value $\tau \to 0$ as the value of time (t) increases. The cosmological term becomes zero when $\beta \to 0$, which gives the perfect fluid solution. From equation (9), $\frac{\dot{T}}{T} = 0$ which confirms that the temperature is constant and independent of time. The implication is that the model is inline with the Boyle's law in kinetic theory. According to the solution the temperature is not the function of time. This solution is totally different from the solutions obtained by various authors.

For $m > -\frac{1}{2}$, the value of β form an increasing arithmetic sequence and the values of b and a form a decreasing arithmetic sequence, where p and γ remain constant. As a result the value of the cosmological term constantly increases with the value of β, while the energy density and the coefficient of the bulk viscosity approach $-\infty$. The relaxation time constantly increases when the value of b decreases at a constant rate. This solution is not acceptable because the value of the coefficient of bulk viscosity (ξ) must be positive $(\xi \geq 0)$. So it violates the entropy equation where the coefficient of the bulk viscosity is real and positive.

For $m < -\frac{1}{2}$, the value of β decreases in a arithmetic sequence, while the value of a and b are non-real. This shows that the term has an impact on the bulk viscosity. The implication is that this cosmological term is not constant, and it mostly depends on the entropy equation as well as the value of the relaxation time and the coefficient of the bulk viscosity.

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