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RESULTS ON AN INTEGRAL INEQUALITY OF THE OPIAL- TYPE

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ABSTRACT

We obtain integral inequalities which are Opial-type inequalities, mainly by using Jensen's inequality for the case of convex function.

KEYWORDS: Integral inequalities, Opial's inequality, Jensen's inequality and convex functions.
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INTRODUCTION

Opial ([8])established the following interesting integral inequality:
Let $x(t) \in C'[0,b]$ be such $x(0) = x(b) = 0$ and $x(t) > 0$ in (0, b), then

$$\int_a^b |x(t)x'(t)|\,dt \le \frac{b}{4}\int_a^b (x'(t))^2\,dt \qquad \text{(1)}$$

where $\frac{b}{4}$ in the best possible constant.

In 1967 Maroni [5] obtained a generalized Opial$'s$ inequality by using $H\ddot{o}$lders inequality with indices « and ¬ . The result obtained is the following:

Theorem 1:

Let p(t) be positive and continuous on $[², ~]$ with $\int_~^² p^{1-«}(t)dt < \infty$, where $« > 1$, x(t) be absolutely function on $[~, ²]$ and x(0)=0. Then, the following inequality holds.

$$\int_~^² |x(t)x'(t)|\,dt \le \frac{1}{2}\left(\int_~^² p^{1-«}(t)dt\right)^{\frac{2}{«}}\left(\int_~^² p(t)\,|x'(t)|^{\ulcorner}\,dt\right)^{\frac{2}{¬}} \qquad \text{(2)}$$

where $\frac{1}{«}+\frac{1}{¬}=1$. Equality holds in (2)iff $c\int_~^² p^{1-«}(s)ds$.

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Calvert [2] also established the following result:

Theorem 2: [2] Assume that

(i) x(t) is absolutely continuous in $[\sim, ^2]$ and $x(\sim) = 0$

(ii) f(t) is continuous, complex-valued, defined in the range of x(t) and for all real for t of the form $t(s) = \int_{\sim}^{s} |x'(u)|\,du : f(|t|)$ for all t and f(t) is real $t > 0$ and is increasing there,

(iii) p(t) is positive, continuous and $\int_{\sim}^{2} p^{1-«}(t)dt < \infty$, where $\frac{1}{«} + \frac{1}{¬} = 1$. Then the following inequality holds.

$$\int_{\sim}^{2} |f(x(t))x'(t)|dt \le F\left[\left(\int_{\sim}^{2} p^{1-«}(t)dt\right)^{\frac{1}{«}}\left(\int_{\sim}^{2} p(t)\,|x'(t)|^{¬}\,dt\right)^{\frac{1}{¬}}\right] \qquad \text{(3)}$$

where $F(t) = \int_{0}^{t} f(t)ds, t > 0$. Equality holds in (3) iff $x(t) = \int_{\sim}^{t} p^{1-«}(s)ds$.

The aim of this paper is to generalize Maroni and Calvert results using Jensen's inequality.

2. Some Adaptations of Jensen's inequalities :

Let ¨ be continuous and convex function and let h(s, t) be a non negative function and ª be non decreasing function. Let $-\infty \le ¶(t) \le ¦(t) < \infty$ and suppose ¨ has a continuous inverse $¨^{-1}$ (which is necessarily concave). Then,

$$¨^{-1}\left(\left|\frac{\int_{¶(t)}^{¦(t)} h(s,t)\,dª(s)}{\int_{¶(t)}^{¦(t)} dª(s)}\right|\right) \le \left(\frac{\int_{¶(t)}^{¦(t)} (¨)^{-1}(|h(s,t)|)\,dª(s)}{\int_{¶(t)}^{¦(t)} dª(s)}\right) \qquad \text{(4)}$$

with the inequality reversed if ¨ is concave. The inequality (4) above is known as Jensen's inequality for convex function. Setting $¨(u) = u^{l-1}, ¶(t) = 0,$ and $¦(t) = t$ in (4), then we obtain

$$(f(t))^{¸} = \left(f\left(\left|\frac{\int_{0}^{t} h(s,t)\,dª(s)}{\int_{0}^{t} dª(s)}\right|\right)\right)^{\frac{1}{¸}} \le \left(\frac{\int_{0}^{t} (|h(s,t)|)^{\frac{1}{l}}\,dª(s)}{\int_{0}^{t} dª(s)}\right). \qquad \text{(5)}$$

3 MAIN RESULT:

Before stating our main result in this section, we shall need the following useful Lemma:

Lemma 1:

Let $x(t), ª(t)$ and $f(u)$ be absolutely continuous and non decreasing functions on $[a,b]$ for $0 \le a \le b < \infty$ with $f(t) > 0$. Let $\sim,\ ¸, k$ and £ be real numbers such that $¢, \ge 0$, $£ \ge 0$ and also Let $P(x)$, and $R(t)$ be non negative and measurable function on $[a,b]$ such that

$$|x'(t)| \times f\left(\left|\int_{0}^{t} x'(t)R(t)\,dª(t)\right|\right) \le ª(t)^{1-¢-£}\, y(t)^{¢}\, R(t)^{-1}\, ª'(t)^{-1}\, y'(t). \qquad \text{(6)}$$

Then, the following inequality holds:

$$\int_{a}^{b} |x'(t)| \times f\left(\int_{0}^{t} |x'(t)|\,dt\right) \le \int_{a}^{b} y(t)^{£}\,dy'(t). \qquad \text{(7)}$$

Proof:

Setting $h(s,t) = x'(t)R(t)$ in (5), we have

$$(f(t))^{£} = \left(f\left(\left| \frac{\int_0^t x'(t)R(t)dª(t)}{\int_0^t dª(t)} \right| \right) \right)^{\frac{1}{£}} \leq \left(\frac{\int_0^t (|x'(t)R(t)|)^{\frac{1}{l}} dª(t)}{\int_0^t dª(t)} \right). \quad \ldots\ldots \quad \textbf{(8)}$$

By setting $f(ª(t)) = ª(t)^{1-¢}$ in (8) yeilds

$$\frac{f(|\int_0^t x'(t)R(t)dª(t)|)}{ª(t)^{1-¢}} \leq \frac{(\int_0^t f(|x'(t)R(t)|)^{\frac{1}{1-¢}} dª(t))^{£}}{ª(t)^{£}}. \quad \ldots\ldots \quad \textbf{(9)}$$

Hence

$$f\left(\left| \int_0^t x'(t)R(t)dª(t) \right| \right) \leq ª(t)^{1-¢-£} \left(\int_0^t f(|x'(t)R(t)|)^{\frac{1}{1-¢}} dª(t) \right)^{£} = ª(t)^{1-¢-£} y(t)^{£}. \quad \ldots\ldots \quad \textbf{(10)}$$

Now let

$$y(t) = \int_0^t f(|x'(t)R(t)|)^{\frac{1}{1-¢}} ª'(t) \quad \ldots\ldots \quad \textbf{(11)}$$

then

$$y'(t) = f(|x'(t)R(t)|)^{\frac{1}{1-¢}} ª'(t). \quad \ldots\ldots \quad \textbf{(12)}$$

That is,

$$y'(t)^{1-¢} = f(|x'(t)R(t)|) ª'(t)^{1-¢}. \quad \ldots\ldots \quad \textbf{(13)}$$

Using the fact that $f(u) = u^{1-¢}$ to have

$$y'(t)^{1-¢} = |x'(t)|^{1-¢} R(t)^{1-¢} ª'(t)^{1-¢}. \quad \ldots\ldots \quad \textbf{(14)}$$

$$|x'(t)| = R(t)^{-1} ª'(t)^{-1} y'(t), \quad \ldots\ldots \quad \textbf{(15)}$$

Combining both (10) and (15) to yeilds, inequality (6) and the proof is complete.

$$|x'(t)| \times f\left(\left| \int_0^t x'(t)R(t)dª(t) \right| \right) \leq ª(t)^{1-¢-£} y(t)^{¢} R(t)^{-1} ª'(t)^{-1} y'(t)$$

Remarks 1 :

By setting $f(u) = u^{1-¢}, R(t) = P(t)^{-\frac{1}{k-1}}, ª'(t)^{\frac{1}{k-1}}, 1-¢ = £$ in Lemma 1 yeilds

$$|x'(t)| \times f\left(\left| \int_0^t x'(t)P(t)^{-\frac{1}{k-1}} P(t)^{\frac{1}{k-1}} dt \right| \right) \leq ª(t)^{1-¢-£} y(t)^{£} \times P(t)^{-\frac{1}{k-1}} P(t)^{\frac{1}{k-1}} y'(t). \quad \ldots\ldots \quad \textbf{(16)}$$

Integrating both sides of inequality(16) over $[a,b]$ with the respect to t, to get

$$\int_a^b |x'(t)| \times f\left(\int_0^t |x'(t)|\,dt\right) \le \int_a^b y(t)^{£}\, y'(t)dt. \quad \text{(17)}$$

That is,

$$\int_a^b |x'(t)| \times f\left(\int_0^t |x'(t)|\,dt\right) \le \int_a^b y(t)^{£}\, dy'(t). \quad \text{(18)}$$

$$\int_a^b |x'(t)| \times f\left(\int_0^t |x'(t)|\,dt\right) \le \frac{y(t)^{£+1}}{£+1} \quad \text{(19)}$$

Setting $\int_0^t |x'(t)|\,dt = x(t)$ By using H ö lders inequality with $\sim$ and we obtain

$$\frac{1}{£+1} y(b)^{£+1} = \frac{1}{£+1}\left(\int_a^b |x'(t)|\,dt\right)^{£+1} = \frac{1}{£+1}\left(\int_a^b R^{-\frac{£+1}{\sim}}(t) R^{\frac{£+1}{}} |x'(t)|(t)dt\right)^{£+1}$$

$$\le \frac{1}{£+1}\left(\int_a^b R^{1-\sim}(t)dt\right)^{\frac{£+1}{\sim}}\left(\int_a^b R(t)|x'(t)| \quad dt\right)^{\frac{£+1}{}} \quad \text{(20)}$$

Combining inequality (19) and (20) to obtain the Opial's Type inequality of the following

$$\int_a^b x'(t) f(x(t))dt \le \frac{1}{£+1}\left(\int_a^b R^{1-\sim}(t)dt\right)^{\frac{£+1}{\sim}}\left(\int_a^b R(t)|x'(t)| \quad dt\right)^{\frac{£+1}{}} \quad \text{(21)}$$

which gives

$$\int_a^b |x'(t)x(t)^{1-¢} \;|\,dt \le \frac{1}{£+1}\left(\int_a^b R^{1-\sim}(t)dt\right)^{\frac{£+1}{\sim}}\left(\int_a^b R(t)|x'(t)| \quad dt\right)^{\frac{£+1}{}} \quad \text{(22)}$$

Remark 2:

For $£ = 0$ in inequality (22) yeilds

$$\int_a^b |x'(t)x(t)^{1-¢} \;|\,dt \le \left(\int_a^b R^{1-\sim}(t)dt\right)^{\frac{1}{\sim}}\left(\int_a^b R(t)|x'(t)| \quad dt\right)^{\frac{1}{}} \quad \text{(23)}$$

Putting $£ = 1$, and $¢ = 0$ in inequality (22) reduces to

$$\int_a^b |x'(t)x(t)|\,dt \le \frac{1}{2}\left(\int_a^b R^{1-\sim}(t)dt\right)^{\frac{2}{\sim}}\left(\int_a^b R(t)|x'(t)| \quad dt\right)^{\frac{2}{}} \quad \text{(24)}$$

which generalizes inequality (2).
If $£ = 1$ and $\sim = 0$ in inequality (22) yeilds

$$\int_a^b |x'(t)x(t)|\,dt \le \frac{1}{2}\left(\int_a^b R(t)|x'(t)| \quad dt\right)^{\frac{2}{}} \quad \text{(25)}$$

If $£ = 0$ in inequality (22) becomes

$$\int_a^b |x'(t)x(t)^{1-¢} \;|\,dt \le \left(\int_a^b R^{1-\sim}(t)dt\right)^{\frac{1}{\sim}}\left(\int_a^b R(t)|x'(t)| \quad dt\right)^{\frac{1}{}} \quad \text{(26)}$$

In inequality (18) if we set $1-¢ = £$ becomes

$$\int_a^b |x'(t)| \times f\left(\int_0^t |x'(t)|\,dt\right) \le \int_a^b y(t)^{1-¢}\, y'(t)dt. \quad \text{(27)}$$

That is,

$$\int_a^b |x'(t)| \times f\left(\int_0^t |x'(t)|\,dt\right) \le \int_a^b f(y(t))dy'(t). \qquad \text{(28)}$$

Getting $\int_0^t |x'(t)|\,dt = x(t)$

Using H$\ddot{o}$lder's inequality with indices $\sim$ and , we have

$$\int_a^b x'(t)dt = \int_a^b R^{-\frac{1}{\sim}}(t)R^{\frac{1}{}}\,|x'(t)|(t)dt \le \left(\int_a^b R^{1-\sim}(t)dt\right)^{\frac{1}{\sim}}\left(\int_a^b R(t)|x'(t)| \quad dt\right)^{\frac{1}{}} \qquad \text{(29)}$$

Combining (28) and (29) to obtain the following inequality

$$\int_a^b |x'(t)| \times f\left(\int_0^t |x'(t)|\,dt\right) \le f\left(\left(\int_a^b R^{1-\sim}(t)dt\right)^{\frac{1}{\sim}}\left(\int_a^b R(t)|x'(t)| \quad dt\right)^{\frac{1}{}}\right) \qquad \text{(30)}$$

that is, inequality that generalizes inequality (3)

$$\int_a^b |x'(t) f(x(t))|\,dt \le f\left(\left(\int_a^b R^{1-\sim}(t)dt\right)^{\frac{1}{\sim}}\left(\int_a^b R(t)|x'(t)| \quad dt\right)^{\frac{1}{}}\right) \qquad \text{(31)}$$

Similarly, we need the following Lemma to obtain a new Opial's type inequality using Jensen's inequality for the case of convex function.

Lemma 2 :
Let $x(t), \text{ª}(t), f(u), R(t), l, k$ and $o \ge 0$ and $° \ge 0$ be as in Lemma 1 such that

$$|x'(t)| \times f\left(\left|\int_0^t x'(t)R(t)d\text{ª}(t)\right|\right) \le y'(t)R(t)^{-1}\text{ª}'(t)^{-1}y(t)^{¢-1} \qquad \text{(32)}$$

Then, the following inequality holds:

$$|x'(t)|\; f\left(\int_0^t |x'(t)|\,dt\right) \le y(t)^{¢-1}y'(t). \qquad \text{(33)}$$

Proof:
The proof is similar to the proof of Lemma 1.
Since $f(u) = u^{¢-1}$, inequality (8) becomes

$$\left(f\left(\left|\frac{\int_0^t x'(t)R(t)d\text{ª}(t)}{\int_0^t d\text{ª}(t)}\right|\right)\right)^{\frac{1}{(¢-1)}} \le \left(\frac{\int_0^t f(|x'(t)R(t)|)^{\frac{1}{¢-1}}d\text{ª}(t)}{\int_0^t d\text{ª}(t)}\right). \qquad \text{(34)}$$

$$f\left(\left|\frac{\int_0^t x'(t)R(t)d\text{ª}(t)}{\int_0^t d\text{ª}(t)}\right|\right) \le \left(\frac{\int_0^t f(|x'(t)R(t)|)^{\frac{1}{¢-1}}d\text{ª}(t)}{\int_0^t d\text{ª}(t)}\right)^{¢-1}. \qquad \text{(35)}$$

$$f\left(\left|\int_0^t x'(t)R(t)d^{ª}(t)\right|\right) \le {}^{ª}(t)^{¢-1+1-¢}\left(\int_0^t f(|x'(t)R(t)|)^{\frac{1}{¢-1}}d^{ª}(t)\right)^{¢-1}. \quad \ldots\ldots\ldots \quad \textbf{(36)}$$

$$f\left(\left|\int_0^t x'(t)R(t)d^{ª}(t)\right|\right) \le \left(\int_0^t f(|x'(t)R(t)|)^{\frac{1}{¢-1}}d^{ª}(t)\right)^{¢-1}. \quad \ldots\ldots\ldots \quad \textbf{(37)}$$

$$y'(t) = |x'(t)|R(t)^{ª}{}'(t) \quad \ldots\ldots\ldots \quad \textbf{(38)}$$

$$y'(t)R(t)^{-1}{}^{ª}{}'(t)^{-1} = |x'(t)| \quad \ldots\ldots\ldots \quad \textbf{(39)}$$

Combining (37) and (39) to have

$$|x'(t)| \times f\left(\left|\int_0^t x'(t)R(t)d^{ª}(t)\right|\right) \le y'(t)R(t)^{-1}{}^{ª}{}'(t)^{-1}y(t)^{¢-1} \quad \ldots\ldots\ldots \quad \textbf{(40)}$$

This completes the proof of the Lemma.
Consider all conditions of Remark 1

$$|x'(t)| \times f\left(\left|\int_0^t x'(t)P(t)^{-\frac{1}{k-1}}P(t)^{\frac{1}{k-1}}\right|\right) \le y'(t)y(t)^{¢-1}P(t)^{-\frac{1}{k-1}}P(t)^{\frac{1}{k-1}}y'(t). \quad \ldots\ldots\ldots \quad \textbf{(41)}$$

$$|x'(t)|\ f\left(\int_0^t |x'(t)|\,dt\right) \le y(t)^{¢-1}y'(t)$$

Then, putting $\int_0^t |x'(t)|\,dt = x(t)$ and integrate both side of inequality above, over $[a,b]$ with the respect to t , yeilds

$$\int_a^b |x'(t)x(t)^{¢-1}|\,dt \le \int_a^b y(t)^{¢-1}y'(t)dt = \frac{1}{¢}y(b)^{¢} \quad \ldots\ldots\ldots \quad \textbf{(42)}$$

$$= \frac{1}{¢}\left(\int_a^b |x'(t)|\,P(t)^{-\frac{1}{o}}P(t)^{\frac{1}{o}}dt\right)^{¢} \quad \ldots\ldots\ldots \quad \textbf{(43)}$$

$$= \frac{1}{¢}\left(\int_a^b P(t)^{1-°}\,dt\right)^{\frac{¢}{°}}\left(\int_a^b |x'(t)|\,P(t)^{\frac{1}{¢}}dt\right)^{\frac{¢}{o}}. \quad \ldots\ldots\ldots \quad \textbf{(44)}$$

We are able to generalized inequality (2), (3) and some remarks on Opial- type inequalities.

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