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# DIRICHLET'S PROBLEM ON A CRACKED TRAPEZIUM

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## ABSTRACT

This paper deals with solving Poisson's equation with conditions on Dirichlet's limits in an isosceles trapezium with two cracks. The large singular finite elements method used gives satisfactory results in all the domain of study. Numerical values obtained are very accurate for the constraint function and its first derivatives except at the ends of cracks where major changes were registered.

**KEYWORDS:** Large elements, finite elements, singularities, cracks.

## INTRODUCTION

Poisson's equation is used in many fields of physics. For example, the study of the elastic deformation of a horizontal membrane applied to distributed load, that of low bar torsion or the flow in a pipe [1-2]. It is also used in many other areas, dissemination of pollutants, heat conduction [3-4], electromagnetism [5], universal gravitation, speed potential, vorticity [6], [etc.]. Solving Dirichlet's problem for Poisson's equation on a domain with cracks is particularly difficult. Indeed, at these points $\pm_i$, the series corresponding to the solution of the homogeneous equation associated with the Poisson's equation are: $\sum_{k=1}^{\infty} a_{ik} r_i^{\frac{k}{2}} \sin \frac{k-}{2}{}_i$ and their first term, which is proportional to $r_1^{\frac{1}{2}}$, shows derivatives tending towards infinity near the end of the crack [7]. Common methods of finite elements or finite differences provide unsatisfactory results when they are used under their standard form. These methods, as shown by various authors [8-13], may be slightly improved if they take the analytical form of the solution near singularities into account. Let's take an isosceles trapezium made of three equilateral triangles, with two cracks and submitted to the torsion. Large singular finite elements method is used to solve this problem. The rationale of the method and its convergence properties are discussed by [14-15].

## MATERIALS AND METHODS

The low bar torsion with polygonal section has been studied using both border collection method Kolodzied and Fraska [16] and Trefftz's integral for complex torsion function, Hassenplug [17]. The equations of the thin bar torsion of the right section Ω are given by Landau [1], Timoshenko and Goodier [18].

$\Delta u(x, y) = -1;\ (x, y) \in \Omega$ ………………………… **(1)**

$u(x, y) = 0;\ (x, y) \in \partial\Omega$ ……………………………….. **(2)**

The domain Ω of $R^2$ is the right section which is a trapezium made of three equilateral triangles with reduced length 2. It bears two cracks. One, from the summit, makes an angle of 60° with the adjacent side to this summit and the other, from the middle of the large basis, makes an angle of 60° with it. Both cracks have the same length (figure 1). The function *u* is a potential of constraints from which we can deduct, by derivation, the non null components (3) and (4) of the constraint tensor at any part of the bar. As to $\partial\Omega$, it is the border of the domain Ω.

$^{2}{}_{xz} = 2G\sim \frac{\partial u}{\partial y}$ ……………………………………… **(3)**

$^{2}{}_{yz} = -2G\sim \frac{\partial u}{\partial x}$ …………………………………… **(4)**

In expressions (1) to (4) $x$ and $y$ are Cartesian coordinates of the point with a domain Ω and $z$ is the direction that makes a tri-rectangular trihedron with the directions of axes $x$ and $y$ of the plane containing $\Omega$. $G$ is the sliding module and $\sim$ the unitary torsion angle.

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The problem then posed is singular because of the four summits of the domain and the two cracks. Considering the study domain allows us noticing that it is not necessary to focus on any symmetry. We must then consider the entire trapezoid domain, which makes the problem more complicated. We will solve it through three steps using large singular finite element method [14]. The large singular finite element method (LSFEM) requires knowledge of the asymptotic solutions of the Poisson equation in the neighborhood of a corner. It is part of the domain decomposition methods. The semi-analytical solutions thus obtained take into account the existence of cracks which correspond to angles of opening 360 °. LSFEM does not eliminate cracks in the field.

**Step 1: Factorization of the Domain**

The first step of the method provides the factorization of the domain into many subdomains as there are singularities. The trapezium is divided into fourteen subdomains involving the use of five pseudo singularities $\pm_2$, $\pm_6$, $\pm_8$, $\pm_{10}$ et $\pm_{14}$ and twenty-five sub-borders as shown in red. It can also be noticed that at the start of the cracks, there are always two merged singularities $\pm_3$ and $\pm_5$, on the one hand, and $\pm_{11}$ and $\pm_{13}$ on the other hand. Five subdomains have 60° angles, including two with 120° ($\pm_3$ and $\pm_9$), the opening of the five pseudo singularities is 180°, while at cracks summits, the opening is 360° (figure 1).

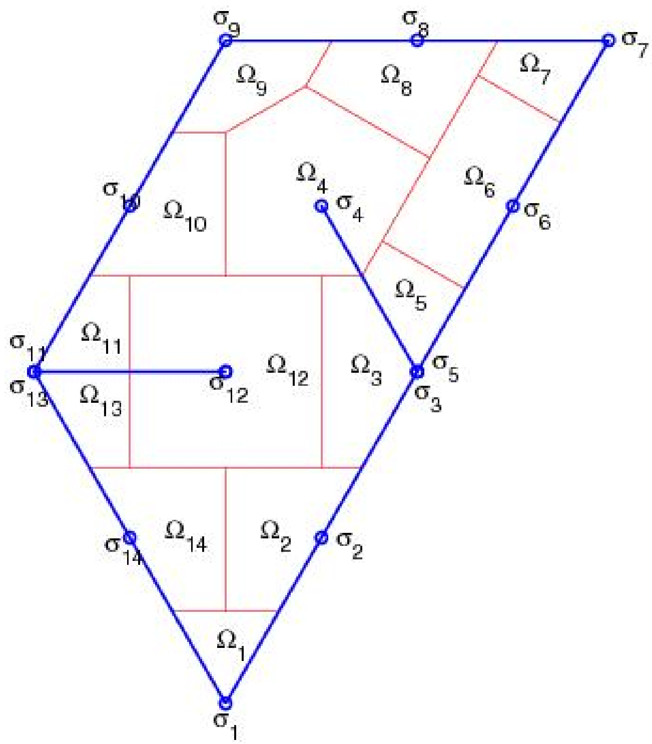


**Figure 1**: Trapezium with two cracks – factorization of the domain into 14 subdomains.

**Step 2: Solving auxiliary problems.**

The second step consists in solving auxiliary problems related to singularities. So, to each subdomain $\Omega_j$ are given an origin $\pm_j$, a singularity, an angle $\sim_j$, the opening angle of the subdomain and a local system of the polar coordinates $(r_j, \bar{}_j)$. It should be that conditions on limits associated with auxiliary problems are all Dirichlet-like. .

For each subdomain $\Omega_j$, we solve the auxiliary problem:

$$\Delta u_j(r_j, \bar{}_j) = -1 \; (r_j, \bar{}_j) \in \Omega_j \quad \text{.............} \quad \textbf{(5)}$$

$$u_j(r_j, 0) = 0 \quad \text{......................................} \quad \textbf{(6)}$$

$$u_j(r_j, \sim_j) = 0 \quad \text{......................................} \quad \textbf{(7)}$$

With $j = 1, \ldots, 14$.

We check that solutions to auxiliary problems related to singularities $\pm_j$ with $j = 1, 2, \ldots, 14$ may be written as follow [14]

$$u_j(r_j, \bar{}_j) = \frac{r_j^2}{4}\left[\frac{\cos 2(\bar{}_j - \sim_j)}{\cos \sim_j} - 1\right] + \sum_{k=1}^{\infty} a_{jk} r_j^{¥_{jk}} \sin ¥_{jk} \bar{}_j \quad \text{......} \quad \textbf{(8)}$$

provided that we give to each subdomain $\Omega_j$, the appropriate value of the opening angle and that we pose $¥_{jk} = \frac{k®}{\sim_j}$ with $j = 1, 2, \ldots, 14$ and $k = 1, 2, \ldots, \infty$.

Practically, we must keep to approximate solutions. The approximation is due, on the one hand, to the fact that developments (8) should be limited to a finite number of terms and, on the other hand, that we must keep, with few exceptions, to an imperfect alignment. The number of coefficients kept in each of the sums has been chosen according to Descloux and Tolley [10] principle which aims at representing approximate solutions with functions that are as uniform as possible. This allows reaching an overall homogeneity for approximate solutions. This can be reached by selecting more terms for subdomains where openings are larger. We decide to keep a number of coefficients $a_{jk}$ proportional to the opening angle $\sim_j$ $(j = 1, 2 \ldots, 14)$ of the extension $\Omega_j^*$ of the subdomain $\Omega_j$. As the highest common divider of all these angles is 60°, we may infer from the proportionality rule that we should keep *N* arbitrary constants in the series related to 60° angles, *2N* for 120° angles, *3N* for 180° angles and 6N at cracks ends. Therefore, the first approximation obtained for N=1 will allow solving a system of 36 equations with 36 unknowns and the nth approximation will derive from a system of 36N unknowns.

**Step 3: Aligning auxiliary solutions**

The third step of the method consists in aligning auxiliary solutions. We limit the series to a number of finite terms as explained above and we align solutions of auxiliary problems according to the continuous least squares. We should find coefficients $a_{kl}$ that allow minimizing the function.

$$I(a_{mn}) = \sum_{i<j} \int_{\Gamma_{ij}} \left[ (u_i(a_{ik}) - u_j(a_{jl}))^2 + \left( \frac{\partial u_i(a_{ik})}{\partial n_i} + \frac{\partial u_j(a_{jl})}{\partial n_j} \right)^2 \right] ds_{ij} \quad \text{…… (9)}$$

These coefficients are a solution to a linear algebraic system with a positive square matrix of 36N equations with 36N unknowns conventionally known as Gauss's normal equations.

$$\frac{\partial I(a_{mn})}{\partial a_{pq}} = 0 \quad \text{……………………………………….. (10)}$$

The accuracy of the approximate solutions is directly related to the quality of the alignment of the auxiliary solutions. It is therefore natural to characterize this accuracy by measuring the imperfections of the continuity conditions. We will use the measurement of the global error defined in (11).

$$I = \sum_{k<l} \frac{1}{S_{kl}} \int_{\Gamma_{kl}} \left[ (u_k - u_l)^2 + \left( \frac{\partial u_k}{\partial n_k} + \frac{\partial u_l}{\partial n_l} \right)^2 \right] ds_{kl} \quad \text{…. (11)}$$

Where $ds_{kl}$ is the arc length of $\Gamma_{kl}$ and $S_{kl}$ its length, $n_k$ and $n_l$ the external norms of the sub-border separating both adjacent subdomains. If the global error is null, the approximate solution obtained coincides with the exact solution.

If the number of coefficients $a_{kl}$ kept increases, the algebraic system to solve become more and more badly conditioned and the matrix of the system may become numerically singular. The numerical conditioning of an algebraic system is conditioned using the number of the spectral condition of its matrix known as conditioning. The conditioning $\chi$(*A*) of the square matrix is the product of the Eucludean norms of *A* and its inverse $A^{-1}$ [19]

$$¡(A) = \|A\| . \|A^{-1}\| \quad \text{……………………………… (12)}$$

## RESULTS AND DISCUSSION

The mode of convergence of the large finite singular elements is exponential as shown in the graph in figure 2 where we see the evolution of the 10-base logarithm of the alignment global error according to the approximation order N (36N being the total number of coefficients $a_{kl}$ kept in the series characterizing the solutions to auxiliary problems). The smallest global error is $2.93 \times 10^{-11}$. It is calculated using N=14. The function u may therefore be calculated with at least 12 precise digits. These hypotheses are confirmed by numerical values entered in table 1.

**Table 1:** Trapezium with cracks – Values of the deflection and its derivatives at sub-borders meeting point $\Gamma_{410}$ , $\Gamma_{412}$ et $\Gamma_{1012}$ .

| N | u | $\frac{\partial u}{\partial x}$ | $\frac{\partial u}{\partial y}$ |
|---|---|---|---|
| 36 | 0.133480794103 | 0.0643109023 | -0.0118640850 |
| 72 | 0.145109744339 | 0.0550931417 | 0.0340923007 |
| 108 | 0.145065081964 | 0.0561144691 | 0.0335274750 |
| 144 | 0.145011269777 | 0.0561745858 | 0.0329989531 |
| 180 | 0.145009778910 | 0.0561012299 | 0.0329803370 |
| 216 | 0.145011392808 | 0.0560971376 | 0.0330016235 |
| 252 | 0.145011470902 | 0.0561000945 | 0.0330027005 |
| 288 | 0.145011438790 | 0.0561004261 | 0.0330020039 |
| 324 | 0.145011436302 | 0.0561002936 | 0.0330019366 |
| 360 | 0.145011437516 | 0.0561002803 | 0.0330019659 |
| 396 | 0.145011437699 | 0.0561002852 | 0.0330019692 |
| 432 | 0.145011437678 | 0.0561002858 | 0.0330019682 |
| 468 | 0.145011437673 | 0.0561002858 | 0.0330019680 |
| 504 | 0.145011437674 | 0.0561002858 | 0.0330019681 |
| 540 | 0.145011437674 | 0.0561002858 | 0.0330019681 |

Besides the global error, the figure 3 shows, depending on 36N, the 10-base logarithm of the conditioning of the matrix of the system. It can be noticed that the matrix-based conditioning worsen when N increases, which shows the degradation of numerical results when N is above 15. However, this is not prejudicial since, to the best of our knowledge, the accuracy obtained based on N=5 is already above what we can get using numerical methods other than LSFEM.

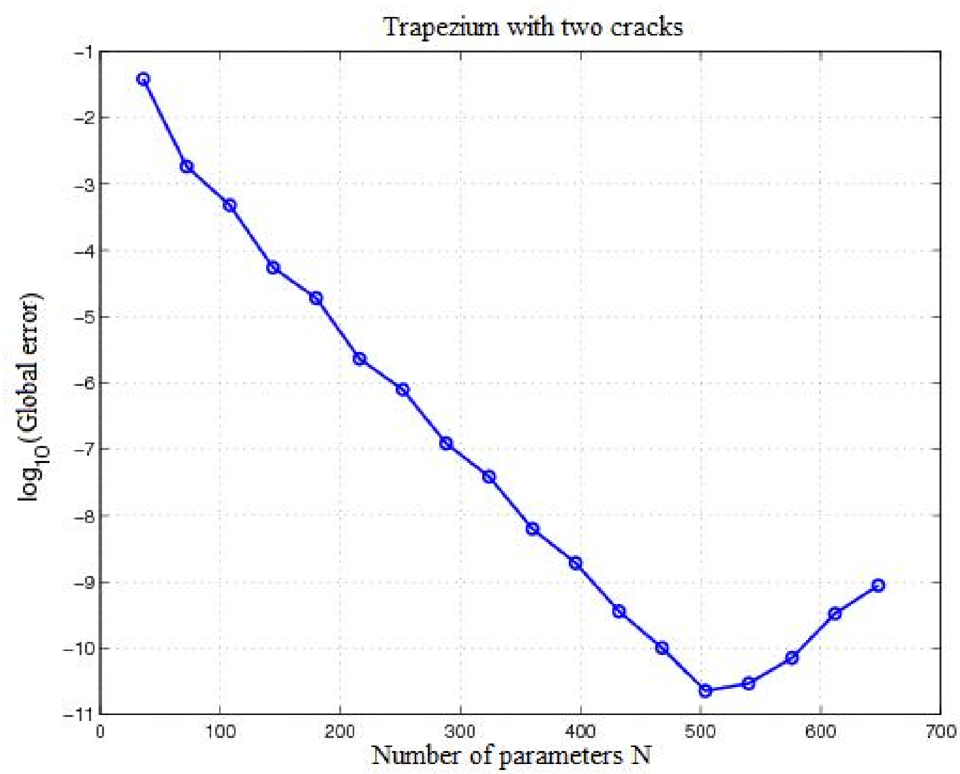


**Figure 2:** Trapezium with two cracks - Evolution of the global error of the alignment according to the total number of coefficients $a_{ik}$ in the developments of auxiliary solutions.

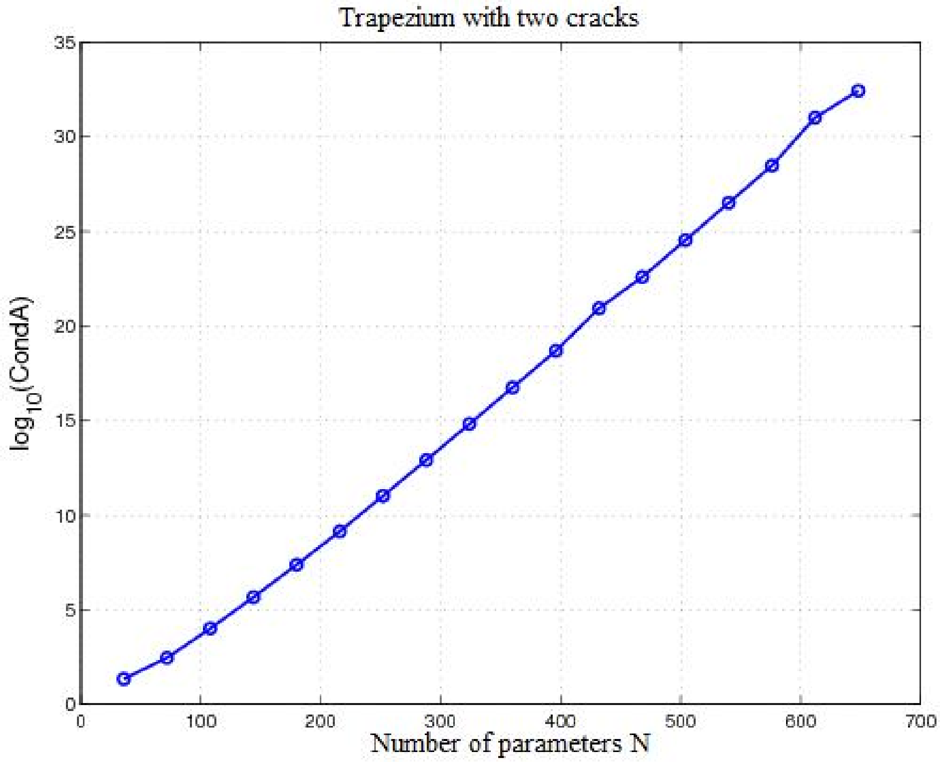


**Figure 3:** Trapezium with two cracks - Evolution of the conditioning of the algebraic system matrix according to the total number of coefficient $a_{ik}$ kept in the developments of auxiliary solutions.

Lastly, the figure 4 shows the plan view and the view in perspective of the function u, while the iso value curves of u, its first partial derivatives and the module of its gradient are schematized in figure 5.

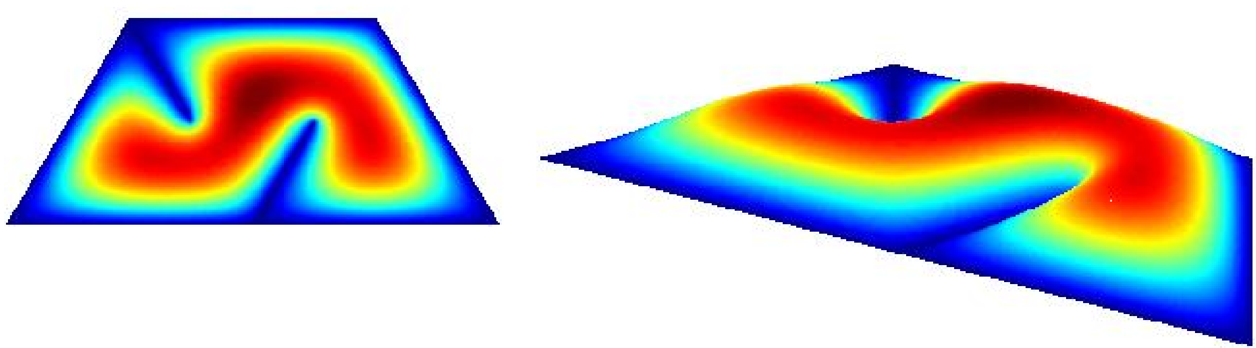

**Figure 4:** Trapezium with two cracks – Plan view and view in perspective of the function *u*.

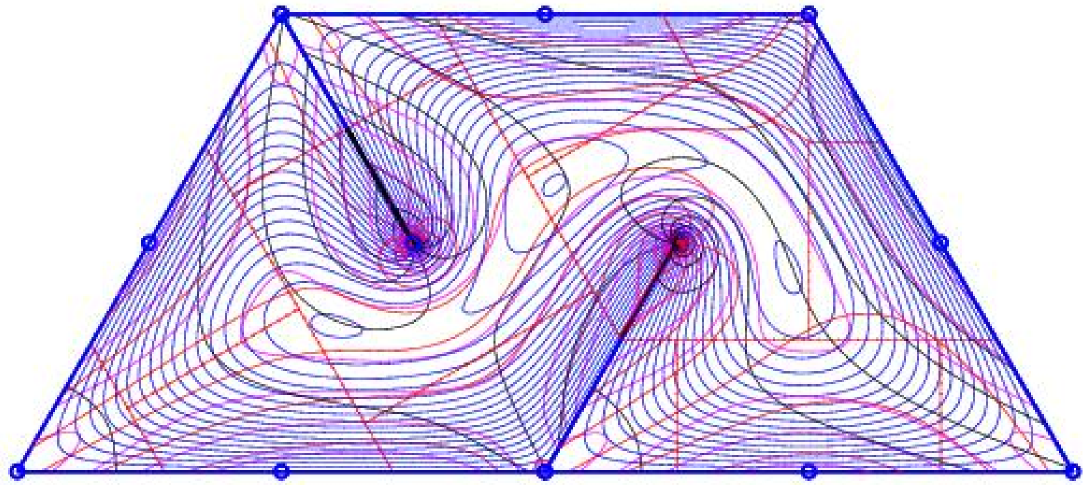

**Figure 5:** Trapezium with two cracks – iso value curves of the function u (blue) of its derivatives $\frac{\partial u}{\partial x}$ (red), $\frac{\partial u}{\partial y}$ (black) and the module of its gradient (magenta).

In this study, we consider the large singular finite element method (LSFEM) using the integration on interelements. The process can be conducted using the boundary collocation method as discussed by Kolodziej and Zielinsski [16]

LSFEM may be used to solve the eigenvalue problem for Laplacian operator Desloux and Tolley [15]. A numerical study of the phenomenon of eigenvalue avoidance is developed by Betcke and Trefeten [20] that reached the same or the best accuracy with a simpler approach.

## CONCLUSION

The study of the trapezium made of three equilateral triangles with two cracks using the large finite elements method provides satisfactory results in the entire domain studied. The method takes the existence of singularities into account by analytically seeking solutions near it; which therefore allows getting, without further formulation, derived magnitude. The convergence of the method is exponential. The smallest global error reaches $2.93\times10^{-11}$. It is calculated using N=14. The function u may therefore be calculated with at least 12 precise digits, while its first partial derivatives must be calculated with at least 10 exact digits. We notice a concentration of constraints at the ends of both cracks.

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