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TIME-INDEPENDENT INVERSE COMPTON SPECTRUM FOR PHOTONS FROM A PLASMA CLOUD OF FINITE EXTENT

J. A. OBU AND E. J. UWAH

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ABSTRACT

The general theoretical aspects of inverse Compton scattering was investigated and an equation for the time-independent inverse Compton spectrum for photons from a plasma cloud of finite extent was derived. This was done by convolving the Kompaneets equation used for describing the evolution of the photon spectrum from a plasma cloud of infinite extent and the distribution of photons over escape time. By looking for a power law solution to the equation so derived, was revealed that the spectrum produced by inverse Compton scattering depend only on the electron temperature and optical depth.

KEY WORDS: Inverse Compton scattering; Plasma cloud; Kompaneets Equation; Optical depth

1 INTRODUCTION

Compton scattering is a scattering phenomenon that causes an exchange of energy between a photon and a charged particle such as an electron.

In astrophysical situations the scattering is repeated (*Comptonization*) and as such, the spectrum of the photons or radiation incident on the plasma becomes distorted. In typical astrophysical collisions, the photon energy is boosted (*inverse Compton scattering*) by a factor~ γ^2 , where γ is the Lorentz factor, (Rybicki and Lightman, 1979).

The Comptonization of photons especially the cosmic microwave background radiation (*called the Sunyaev-Zeldovich effect*) by hot gas in clusters, imprints unique spectral signatures that can be used as important astrophysical and cosmological probes. So, it is worthy to study how the photon spectrum evolves for specific geometries of the electron cloud. One of such studies led to the formulation of an equation now called the Kompaneets equation whose solution describes the time evolution of the spectrum of photons resulting from the combined effect of repeated Compton scattering and of photon production processes, namely radiative Compton (RC), bremsstrahlung (Rybicki and Lightman, 1979), plus other possible photon emission/absorption contributions in the limit of an infinitely thick thermal bath of electron cloud.

2 FINITE EXTENT CLOUD SOLUTION TO THE KOMPANEETS EQUATION

The Kompaneets equation (Kompaneets, 1957; Birkinshaw, 1999) which is a parabolic partial differential equation is given by

$$\frac{\partial n(x,y)}{\partial y} = \frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left(\frac{\partial n(x,y)}{\partial x} + n(x,y) + n(x,y)^2\right)\right] \qquad 1$$

where $x = \frac{\hbar\omega}{k_B T_e}$ is the dimensionless photon frequency, *y* is the Kompaneets *y*-parameter defined by $y \equiv n_e \sigma_T c \Theta \tau$, n_e is the electron number density, σ_T is the Thompson cross section, *c* is the speed of light; $\Theta \equiv kT_e / m_e c^2$ is the dimensionless electron temperature, with T_e the electron temperature and τ is the optical depth of the cloud. For a stationary case the *y*-parameter is defined as the product $\Theta\tau$, and is a measure of the total amount of energy transferred from the electrons to the photons. The electron density n_e is assumed to be constant.

A more general solution of the equation is given as

$$n(x,y) = \frac{1}{\sqrt{4\pi y}}\int_{-\infty}^{\infty} n_0(z')\exp\left\{-\frac{(\ln(x) = 3y - z')^2}{4y}\right\}dz', \qquad 2$$

where $z' = \ln(x')$. This solution to Kompaneets' equation was first obtained by Sunyaev and Zeldovich (1970).

Since cosmological plasma clouds are finite in extent, it is expedient to solve the Kompaneets equation for such clouds so as to appreciate how the photon spectrum evolves on collision with the electrons in such clouds. We

J. A. Obu, Department of Physics, University of Calabar, P.M.B.1115, Calabar
E. J. Uwah, Department of Physics, University of Calabar, P.M.B.1115, Calabar

use the idea that if one knew how many of the photons had escaped from a slab between time *t* and *t* + *dt*, that is, the distribution of photons over the escape time from the slab, one could convolve this with eqn. 2 and determine the spectrum of escaping photons in a cloud with slab-like geometry. By considering the one-dimensional diffusion equation

$$\frac{\partial I(\tau,u)}{\partial u} = \frac{1}{3}\frac{\partial^2 I(\tau,u)}{\partial \tau^2} \qquad 3$$

where $u = y/\Theta$ is a dimensionless time parameter and $\tau = n_e \sigma_T r$ where the variable *r* measures the distance from the slab mid-plane. *I* is the angle-averaged intensity, where the photon occupation number is related to the intensity *I* by $I = bnx^3$. It is assumed that one face of the slab is along the plane $\tau = 0$ and the other face is at $\tau = 2\tau_0$ where $\tau_0 = n_e \sigma_T h$ with h the half-thickness of the slab.

Equation (3) is separable, so if one looks for solutions of the form $X(\tau)R(u)$ we obtain

$$I(\tau,u) = \sum_{n=1}^{\infty} c_n X_n(\tau) \exp\left\{-\frac{(\lambda_n)^2}{3}u\right\} \qquad 4$$

In general, the boundary condition of the problem is used to determine the separation constant λ_n, while the initial distribution $n(x,0)$ is used to determine the coefficient c_n. For an arbitrary injection of photons, there may be infinitely many terms in the series solution given by eqn. 4. However, because λ_n increases with *n*, after some time the higher terms or modes will have died away compared to the first term or 'fundamental' mode. Therefore, the spatial distribution of photons approaches the distribution given by the first eigenfunction $X_1(\tau)$ of the spatial equation. That is, at 'latter times' one has

$$I(\tau,u) \to c_1 X_1(\tau)\exp(-\beta u) \qquad 5$$

where $\beta \equiv \lambda_1^2/3$.

The distribution of photons over the escape time is simply the intensity at the boundary of the slab as a function of time *u*, and this can be given by eqn. 5. A normalization condition is also required for the fraction of photons that have escaped. Using eqn. 5 we write

$$\begin{aligned} P(u) &\equiv \frac{I(0,u)}{\int_o^\infty I(0,u)\,du} \\ &= \frac{c_1 X_1(0)\exp(-\beta u)}{c_i X_1(0)/\beta} \\ &= \beta \exp(-\beta u). \end{aligned} \qquad 6$$

The function *P(u)* is the required distribution of photons over the escape time. As discussed above, if the function *P(u)* is known it can be convolved with the solution to the Kompaneets' equation to give the spectrum from a plasma cloud of finite extent, i.e.

$$N(x) = \int_o^\infty n(x,u)P(u)\,du \qquad 7$$

Therefore the parameter β must be determined and this is done by appealing to the boundary condition of the problem.

For the spatial part of the problem one has the equation

$$X_n''(\tau) + \lambda_n^2 X_n(\tau) = 0 \qquad 8$$

which has the orthogonal solutions

$$X_n = \cos[\lambda_n(\tau_o - \tau)] + \sin[\lambda_n(\tau_o - \tau)] \qquad 9$$

If one appeals to symmetry about the mid-plane of the disk and discards the sine solution, we get

$$X_n = \cos[\lambda_n(\tau_o - \tau)] \qquad 10$$

This equation is then used with the boundary condition to give λ_n (or equivalently β). The boundary condition from diffusion theory is (Zweifel, 1973)

$$\frac{1}{2} = -D\frac{\partial I}{\partial \tau} \qquad 11$$

where the diffusion constant *D* in this case equals 1/3, from eqn. 3. This boundary condition implies that the number of escaping photons (the right hand side) is equal to half the number of photons at the boundary (the left hand side). Presumably the other half of the photons is directed back into the cloud due to the assumed isotropy of the photon distribution. In addition, this condition holds that no photons are incident on the cloud from the outside.

The boundary condition given in eqn. 11 presents difficulties such as, the extent to which the intensity goes to zero and as such an easier approach was formulated by considering the linear extrapolation

$$I(\tau) = I(0)(1 - k\tau) \qquad 12$$

Substituting this into the boundary condition (eqn. 11) and solving for *k* gives *k* = 3/2. So the flux outside the surface of the cloud can be taken to obey

$$I(\tau) = I(0)(1-(3/2)\tau) \tag{13}$$

This implies that the intensity goes to zero at a distance $\tau = 2/3$ outside the boundary. This gives a new and simple boundary condition. Evaluating the intensity at the point outside the boundary (the so-called extrapolated length) and equating to zero, one obtains

$$X_n(-2/3) = 0 \Rightarrow \cos\left[\lambda_n(\tau_o + 2/3)\right] = 0 \tag{14}$$

so that

$$\lambda_n = \frac{(2n-1)\pi}{2(\tau_0 + 2/3)} \Rightarrow \beta = \frac{\lambda_1^2}{3} = \frac{\pi^2}{12(\tau_o + 2/3)^2} \tag{15}$$

Now that β is known, everything about the distribution of photons over the escape time *P(u)* is known. It is also seen that the spatial transport depends only on the optical depth τ_o. This analysis can be repeated for spherical and cylindrical geometries, the principal difference being that value of β (obtained from the boundary condition) is different.

3 THE STATIONARY EQUATION

Knowing the distribution of photons over the escape time *P(u)*, and given eqn. 7, the spectrum from a finite cloud can be found by performing the integration. However, more insight can be gained by using the same equations to derive another equation for the spectrum of a finite cloud; one that is stationary in time. To do this, we construct an equation for the time-independent spectrum *N(x)* such that

$$N(x) = \int_o^\infty n(x,u)P(u)du \tag{16}$$

This is done by substituting both the Kompaneets expression for *n(x,u)* and the equation for *P(u)* (eqn. 6). Dropping the $n(x,u)^2$ term in Kompaneets equation, multiplying each side by $P(u)$ and then integrating gives

$$\int_o^\infty \beta \frac{\partial n}{\partial u}\exp(-\beta u)du = \frac{\Theta}{x^2}\frac{\partial}{\partial x}x^4\left\{\int_o^\infty \beta n \exp(-\beta u)du + \frac{\partial}{\partial x}\int_o^\infty \beta n\exp(-\beta u)du\right\}$$

$$= \frac{\theta}{x^2}\frac{\partial}{\partial x}\left[x^4\left(\frac{\partial N(x)}{\partial x} + N(x)\right)\right] \tag{17}$$

The left hand side equals

$$\left[\beta n\exp(-\beta u)\right]_o^\infty + \int_o^\infty \beta^2 n\exp(-\beta u)du$$

$$= 0 - \beta n(x,0) + \beta N(x) \tag{18}$$

$$= -\frac{\beta f(x)}{x^3} + \beta N(x)$$

where f(x) is a source that is equal to $n(x,0)x^3$. So the equation becomes

$$\frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left(\frac{\partial N(x)}{\partial x} + N(x)\right)\right] = \gamma N(x) - \gamma\frac{f(x)}{x^3} \tag{19}$$

where $\gamma \equiv \beta/\Theta$.

The left hand side is identical to the Kompaneets equation. The first term on the right hands side describes the escape of photons from the plasma cloud, while the second term describes the injection of photons due to a source, f(x). The value of γ is given by the spatial diffusion problem.

By looking for a power law solution to the stationary equation of the form

$$N(x) = ax^{-\alpha(\alpha+3)} \tag{20}$$

where α is the energy spectral index, it is found that

$$\alpha(\alpha+3) = \gamma = \frac{\pi^2}{12\Theta(\tau_0 + 2/3)^2} \tag{21}$$

for a slab geometry (using the value of β found earlier).Solving this quadratic equation gives

$$\alpha = -\frac{3}{2} \pm \sqrt{\frac{9}{4} + \frac{\pi^2}{12\Theta(\tau_o + 2/3)^2}} \tag{22}$$

This equation relates the energy spectral index α to the temperature and optical depth of the cloud. From the equation, we see that the spectrum produced by inverse Compton scattering depends only on the electron temperature and optical depth. It is also seen that the spectral slope is inversely proportional to Θ and τ_o.

4 CONCLUSION

The Comptonization process for photons undergoing repeated scattering in a cloud of finite slab-like geometry was investigated and an expression for the time-independent spectrum $N(x)$ consequently derived. It was shown, after appealing to a power law solution, that the spectrum produced by inverse Compton scattering depends only on the electron temperature and optical depth. This analysis can be extended to spherical and cylindrical geometries; with the principal difference being that the value of β (obtained from the boundary condition) is different.

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