DOI: https://dx.doi.org/10.4314/gjpas.v30i4.13

GLOBAL JOURNAL OF PURE AND APPLIED SCIENCES VOL. 30, 2024: 551-558
COPYRIGHT© BACHUDO SCIENCE CO. LTD PRINTED IN NIGERIA ISSN 1118 – 0579, e-ISSN: 2992 - 4464
www.globaljournalseries.com.ng, Email: globaljournalseries@gmail.com

A CLASS OF POSITIVE DEFINITE SPHERICAL FUNCTIONS ON THE EUCLIDEAN MOTION GROUPS

U. E. EDEKE, R. D. ARIYO AND O. C. DADA

(Received 23 August 2024; Revision Accepted 18 September 2024)

ABSTRACT

Let G be the Euclidean motion group $\mathrm{G} = \mathbb{R}^n \rtimes \mathrm{SO(n)}$ realized as the semi-direct product of $\mathbb{R}^n$ and $\mathrm{SO(n)}$. Let $\mathrm{K} = \mathrm{SO(n)}$ be a compact subgroup of G. The set of positive definite spherical functions on G is studied. Among other things, a result of Bochner which characterizes a K- bi-invariant positive definite spherical function in any locally compact group, is extended to the Gelfand pair $(\mathbb{R}^n \rtimes \mathrm{SO(n)}, \mathrm{SO(n)})$.

KEYWORDS: Euclidean motion group, Gelfand pair, positive definite functions, spherical function, Bochner's theorem.
2020 AMS Subject Classification. 43A70, 43A90.

INTRODUCTION

Positive definite functions play significant role in analyzing the structure of convolution algebras and in the theorem of Bochner. This theorem, which is generally referred to as Bochner's theorem, characterizes positive definite functions on $\mathbb{R}^n$ as Fourier transforms of non-negative Borel measures; and has a generalizability for Locally Compact Abelian (LCA) groups and for Gelfand pairs. The class of positive definite functions is closed under addition and multiplication with nonnegative constant. It has the structure of a cone and the cone is closed in the topology of pointwise convergence. It is also worth noting that every positive definite function ϕ on G is bounded, this is because $|\phi(g)| \leq \phi(e), \forall g \in G$, where e is the identity element of G [15].

The focus of this paper is to state and prove Bochner's theorem for spherical functions on the Euclidean motion group. That is, characterizing positive definite K- bi- invariant functions on the Euclidean motion group is the focus of this paper. The spherical functions on (G,K) are identified with radial functions on the Euclidean plane([2], [7]). Specifically, the spherical functions for the Gelfand pair (M(n), SO(n)) are the spherical Bessel functions of order n ([6],[8]). The set of positive definite spherical functions, denoted by $(G,K)_+^\tau$, are identified with the positive real line.

This paper is organized as follows. In section two, preliminaries on Semi Direct Product group is given and is narrowed to the Euclidean motion group which is an example of a semi-direct product group; its integration, Lie algebra and representation are discussed. Spherical functions on the motion group arising from its Gelfand pairs (G,K) are also discussed. Finally, a discussion on a set of spherical functions that are positive definite is presented. In section three, a result of Bochner, among other things, that characterizes positive definite spherical functions is stated and proved for the Euclidean motion group

U. E. Edeke, Department of Mathematics, University of Calabar, Calabar, Nigeria.
R. D. Ariyo, Department of Mathematics, University of Ilesa, Ilesa, Nigeria.
O. C. Dada, Department of Mathematics, Redeemer University, Ede, Nigeria.

Preliminaries

Definition. A semi-direct product of two groups N and K is a group G such that ([12],[9]) $G = N \rtimes K$.

As a set, G may be written as

$$G = N \times K$$

and any element $g \in G$ can uniquely be written as $g = nk$ for all $n \in N$ and $k \in K$. N is a normal subgroup of G. This condition is usually written as $N \triangleleft G$. There is a homomorphism defined as

$$\phi: k \in K \to \phi_k \in Aut(N)$$

such that the group law in $N \rtimes K$ is defined by

$$(n_1k_1)(n_2k_2) = n_1k_1n_2k_2 = n_1\phi_{k_1}(n_2)k_1k_2$$

Aut (N) denotes the automorphism group of N to itself. For $k \in K$ the automorphism ϕ_k is given by the conjugation

$$\phi_k: N \to N$$

defined by

$$\phi_k(n) = knk^{-1}$$

The next theorem shows that ϕ_k is a homomorphism.

Theorem. The map $k \mapsto \phi_k$ is given as a homomorphism $\phi: k \to Aut(N)$

Proof. For any $k_1, k_2 \in K$, it is shown that $\phi_{k_1}\phi_{k_2} = \phi_{k_1k_2}$. Since $\phi_{k_1}\phi_{k_2}$ and $\phi_{k_1k_2}$ are functions. Next we show that the two functions are equal. To this end, it will be shown that they take the same value on any element n of N , that is

$$\phi_{k_1}\phi_{k_2} = \phi_{k_1k_2}$$

for any $n \in N$. Now, since

$$\phi_k: N \to N$$

is defined as

$$\phi_k(n) = knk^{-1}$$

then

$$\begin{aligned}\phi_{k_1}\phi_{k_2}(n) &= \phi_{k_1}(k_2nk_2^{-1}) \\ &= k_1(K_2nk_2^{-1})k_1^{-1} \\ &= k_1k_2nk_2^{-1}k_1^{-1}\end{aligned}$$

Similarly,

$$\begin{aligned}\phi_{k_1k_2}(n) &= k_1k_2(n)(k_1k_2)^{-1} \\ &= k_1k_2nk_2^{-1}k_1^{-1}\end{aligned}$$

Therefore, $\phi_{k_1}\phi_{k_2} = \phi_{k_1k_2}$ from the last two equations. The next theorem shows that given two groups N and K and a homomorphism, a semidirect product group can be constructed.

Theorem. Given two groups N and K and a homomorphism $K \to Aut(N)$, there is a semidirect product group G based on this information. The group can be constructed as follows. The underlying set of G is the set of pairs (n, k) with $n \in N$ and $k \in K$. The multiplication of this set is given by the rule

$$(n, k)(n', k'_{\prime}) = (n\phi_k(n'), kk')$$

The identity element is $(1,1)$ and the inverse is given by

$$(n, k)^{-1} = (\phi_{k^{-1}}(n^{-1}), k^{-1})$$

Let N and K be two groups, where N is abelian and K is compact. Let $G = N \rtimes K$ be the semi direct product of N and K. The underlying manifold of G is $N \times K$. Therefore, the Haar measure of G is the product of the Lebesque measure on N and the Haar measure on K. That is, if dn is the Lebesque measure of N and dk is the Haar measure of K, the Haar measure of G is $dg = dndk$. The next theorem summarizes the existence of Haar measure on a semidirect product group.

Theorem. Let G be a semidirect product group and let $\phi: K \to \mathrm{Aut}(N)$ be a homomorphism. If $g = (n, k) \in G$, then

$$d\mu_G(g) = \|\phi(k)\|^{-1} d\mu_N(n) d\mu_K(k)$$

and

$$\triangle_G(g) = \|\phi(k)\|^{-1} \triangle_N(n) \triangle_K(k)$$

proof. To see the formula for μ_G, we compute

$$\begin{aligned}
&\int_K \left\{\int_N f\big((n', k')(n, k) d\mu_N(n)\big\}\right) \|\phi(k)\|^{-1} d\mu_k(k) \\
&= \int_K \left\{\int_N f(n'\phi(k')n, k'k) d\mu_N(n)\right\} \|\phi(k)\|^{-1} d\mu_K(k) \\
&= \int_K \left\{\int_N f(n'\phi(k')n, k) d\mu_N(n)\right\} \|\phi(k'^{-1}k)\|^{-1} d\mu_K(k) \\
&= \int_K \left\{\int_N f(\phi(k')(\phi(k'^{-1})(n')n), k) d\mu_N(n)\right\} \|\phi(k'^{-1}k)\|^{-1} d\mu_K(k) \\
&= \int_K \left\{\int_N \left(f(\phi(k'^{-1})(n')n), k\right) d\mu_N(n)\right\} \|\phi(k)\|^{-1} d\mu_K(k) \\
&= \int_K \left\{\int_N f(n, k) d\mu_N(n)\right\} \|\phi(k)\|^{-1} d\mu_K(k).
\end{aligned}$$

The formula for $\triangle_G$ can be used to integrate functions of the form

$$f(n, k) = \nu(n)\nu(k) \tag{1}$$

To this end,

$$\begin{aligned}
\int_G f(g) d\mu_G(g) &= \int_K \left\{\int_N f((n, k)^{-1}) \triangle_G((n, k)^{-1}) d\mu_N(n)\right\} \|\phi(k)\|^{-1} d\mu_K(k) \\
&= \int_K \left\{\int_N f(\phi(k)^{-1}(n^{-1}), k^{-1}) \triangle_G\big((\phi(k)^{-1}(n^{-1}), k^{-1}) d\mu_N(n)\big\} \|\phi(k)\|^{-1} d\mu_K(k) \\
&= \int_K \left\{\int_N \nu\big(\phi(k)^{-1}(n^{-1})\big) \Delta_G\big((\phi(k)^{-1}(n^{-1}), k^{-1}) d\mu_N(n)\big\} \nu(k^{-1}) \|\phi(k)\|^{-1} d\mu_K(k) \\
&= \int_K \left\{\int_N \nu(n^{-1} \triangle_G(n^{-1}, k^{-1}) d\mu_N(n)\} \nu(k^{-1})\right\} d\mu_K(k) \\
&= \int_K \left\{\int_N \nu(n) \triangle_G(n, k) \triangle_N(n^{-1}) d\mu_N(n)\right\} \nu(k) \Delta_K(k^{-1}) d\mu_K(k) \\
&= \int_K \int_N f(n, k) \triangle_G(n, k) \triangle_N(n^{-1}) \Delta_K(k^{-1}) d\mu_N(n) d\mu_K(k)
\end{aligned}$$

Thus

$$\triangle_G(n, k) \triangle_N(n^{-1}) \triangle_K(k^{-1}) = \|\phi(k)\|^{-1}$$

Given $G = N \rtimes K$, as mentioned earlier, the next result shows that the Haar measure on G is the product of the Haar measure on N and the Haar measure on K.

Theorem. Let $G = N \rtimes_\phi K$. Let m_G, m_N and m_K denote their respective right Haar measures. If $g = (n, k)$ then

$$dm_G(g) = dm_N(n) dm_K(k)$$

That is

$$\int_G f(n, k) dm_G = \int_N \int_K f(n, k) dm_K(k) dm_N(n)$$

A general description of the representation of a semi-direct product group is given as follows. Let $G = N \rtimes_\phi K$ be the semi direct product of a commutative group N and its automorphism group K. Let χ be the scalar function $\chi(n)$, or a characteristic of $k, n \in N$, such that

$$\chi(n_1 + n_2) = \chi(n_1)\chi(n_2)$$

For each automorphism a of N there is an automorphism $\hat{a}$ of χ, defined by the formula

$$\hat{a}\chi(b) = \chi(a^{-1}b)$$

where $\hat{a}_1\hat{a}_2 = (\widehat{a_1, a_2})$. Therefore, K is isomorphic to the group of automorphism $\hat{k}$ for $\chi \cdot \chi$ is decomposed into transitivity classes for transformation $\hat{K}$. Let Φ be one of those classes and let $f(\Phi)$ be a function given on Φ. With each element $g = (b, a)$ of G, an operator $T(g)$ is defined by the formula

$$T(g)f(\varphi) = \varphi(b)f(\hat{a}^{-1}\varphi)$$

Since, for $g_1(b_1, a_1), g_2(b_2, a_2)$, we have

$$\begin{aligned} T(g_1)T(g_2)f(\varphi) &= T(g_1)\varphi(b_2)f(\hat{a}_2^{-1}) \\ &= \varphi(b_1)\hat{a}_1^{-1}\varphi(b_2)f_2(\hat{a}_2^{-1}\hat{a}_1^{-1}\varphi) \\ &= \varphi(b_1 + a_1b_2)f[(\hat{a}_1\hat{a}_2)^{-1}\varphi] \end{aligned}$$

and

$$(b_1, a_1)(b_2, a_2) = (b_1 + a_1b_2, a_1a_2)$$

Therefore, $T(g)$ is a representation of G in the space of function on Φ.
The Euclidean Motion group is a Locally Compact topological Group [1]. It is a semi direct product of the additive group $\mathbb{R}^n$ and the Orthogonal group $O(n)$. There is also Special Euclidean Motion group, often represented as $SE(n)$. This group is the semi direct product of $\mathbb{R}^n$ and the special orthogonal group, $SO(n)$. That is,

$$SE(n) = \mathbb{R}^n \rtimes SO(n)$$

where $SO(n) = SL(n) \cap O(n)$

This group is also called group of transformation of the Euclidean plane, which preserves the distance between points and do not change the orientation of the plane [5]. Our focus is on the special Euclidean group which, henceforth, shall be represented by $M(n)$, for $n = 2$. Elements of $M(2)$ are given by $g = (a, A) \in M(2)$, where $A \in SO(2)$ and $a \in \mathbb{R}^2$. For any $g = (a, A)$ and $h = (r, R)$, the group law of $M(2)$ is given as

$$gh = (a, A)(r, R) = (a + Ar, AR)$$

and the inverse g^{-1} is given by

$$g^{-1} = (-A^T a, A^T)$$

Elements of $M(2)$ can be identified as a 3×3 homogeneous transformation matrix of the form

$$H(g) = \begin{pmatrix} A & a \\ 0^T & 1 \end{pmatrix}$$

where $A \in SO(2)$ and $0^T = (0,0), H(g)H(h) = H(g \circ h)$ and $H(g^{-1}) = H^{-1}(g)$. The mapping $g \mapsto H(g)$ is an isomorphism between $M(2)$ and the set of homogeneous transformation matrices. That is,

$$M(2) = \mathbb{R}^2 \rtimes SO(2) \cong B \subset GL(3, \mathbb{R}),$$

where B is a closed subgroup of $GL(3, \mathbb{R})$. Let $\mathfrak{D}$ be the space of infinitely differentiable functions $f(x)$ on the circle $x_1^2 + x_2^2 = 1$. Let R be a fixed complex number. Each element $g(a, \alpha) \in M(2)$, where $a \in \mathbb{R}^2$ and $\alpha \in SO(2) \cong [0,2\pi] \cong \mathbb{T}$, is associated an operator $T_R(g)$ that transforms $f(x)$ into the function

$$T_R(g)f(x) = e^{R(a,x)}f(x_{-\alpha}) \quad (2)$$

where $x_{-\alpha}$ is the vector into which x is transformed under a rotation by an angle $-\alpha$ and $(a, x) = a_1x_1 + a_2x_2$ [11]. $T_R(g)$ is an irreducible unitary representation of $M(2)$ [10]

MAIN RESULT

Let G be a locally compact group, let K be a compact subgroup of G and let $L^1(G)$ be a convolution algebra of G. Some basic definitions concerning Gelfand pair and spherical functions are required before the main result.

Definition ([2]). A function $f: G \to \mathbb{C}$ is said to be bi-invariant under K if it is constant on double coset of K . That is, if $f(k_1gk_2) = f(g) \forall k_1, k_2 \in K$ and $\forall g \in G$

Let $C_c(G)^k$ (resp. $L^1(G)^k$) be the set of continuous compactly supported (resp. L^1) functions that are bi-invariant under K. $C_c(G)^k$ (resp. $L^1(G)^k$) is a subalgebra of $C_c(G)$ (resp. $L^1(G)$)

3.2 Definition [2]. The pair (G,K) is called a Gelfand pair if $L^1(G)^k$ in def. 3.1 above is a commutative algebra. In another formulation, the pair (G,K) is a called a Gelfand pair if the Banach * - algebra $L^1(K \setminus G/K)$ of a K-bi-invariant integrable functions on G is commutative.

Given a function $\varphi \in C(G)$ (not necessarily compactly supported), we consider the linear functional

$$\chi_\varphi: C_c(G) \to \mathbb{C}$$

defined as

$$\chi_\varphi(f) = \int_G f(x)\varphi(x^{-1})dx$$

This functional is what shall be used in defining spherical functions for the pair (G,K) as we make progress. The following proposition provides a condition for a certain pair (G,K) to be a Gelfand pair.

Proposition ([7], prop. 6.1.3). Let G be a locally compact group and K a compact subgroup of G. Assume there exists a continuous involutive automorphism ϕ of G such that

$$\phi(x) \in Kx^{-1}K$$

for all $x \in G$. Then (G, K) is a Gelfand pair.
In a different formulation, functions on G that are bi-invariant under K can (by restriction) be identified with functions on $\mathbb{R}^n$ satisfying

$$f(k \cdot g) = f(g)$$

These functions are referred to as the radial functions and the convolution product of two functions of such corresponds with the ordinary convolution product on $\mathbb{R}^n$. This shows that the algebra of bi-invariant functions on G, denoted as $C_c(G)^k$ is a commutative convolution algebra.

Definition $[2]$: A spherical function

$$\varphi: G \to \mathbb{C}$$

for the Gelfand pair (G, K) is a k-bi-invariant C^∞ - function on K with $\varphi(e) = 1$ and satisfy one of the following three equivalent conditions

1. $\int_K \varphi(xKy)dx = \varphi(x)\varphi(y)$
2. $f \to \int_G f(g)\overline{\varphi(g)}dg$ is a homomorphism of $C_c(K \setminus G/K)$ into $\mathbb{C}$
3. φ is an eigen function of each $D \in \mathfrak{D}(G/K) \cdot \mathfrak{D}(G/K)$ is the algebra of k -invariant differential operators on G/K

Also, a function $\varphi \in C(G), \varphi \neq 0$, is said to be spherical if it is bi-invariant under k and χ_φ is a character of $C_c(G)^K$. That is, $\forall f, g \in C_c(G)^K$

$$\chi_\varphi(f * g) = \chi_\varphi(f) \cdot \chi_\varphi(g)$$

Alternatively, let G be a connected and simply connected unimodular solvable Lie group with Haar measure μ, and let K be a connected compact group acting on G as automorphism. A bounded continuous function φ on G is called a K -spherical function if for $x, y \in G$ the following holds

$$\int_K \varphi(x(ky))dk = \varphi(x)\varphi(y)$$

and $\varphi(1_G) = 1$, where dk is normalized. The Banach space $L^1(G)$ of integrable functions on G has a structure of Banach * - algebra with convolution and involution defined respectively by

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)d\mu(y)$$

and

$$f^*(x) = \overline{f(x^{-1})}$$

K acts on $L^1(G)$ as automorphism by $f^k(c) = f(k^{-1}x)$ for $x \in G, k \in K$. Also, $L^1(G)^k$ is a closed * - subalgebra of $L^1(G)$. For a bounded continuous function φ on G, we define a linear functional λ_φ on $L^1(G)$ by

$$\lambda_\varphi(f) = \int_G f(x)\varphi(x)d\mu(x)$$

φ is called K-spherical if and only if λ_φ is multiplicative on $L^1(G)^k$. This multiplicity is defined as

$$\lambda_\varphi(f * g) = \lambda_\varphi(f)_\varphi(g)$$

for $f, g \in L^1(G)^k$

The Laplace operator for $M(2)$ is a Laplacian in cylindrical polar coordinate in three dimensional. It is given as

$$\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\partial^2\varphi}{\partial z^2} = 0 \tag{3}$$

The solution of the radial part of (3) is a Bessel function of order zero. It is given below

$$J_\lambda(mr) = \Gamma(1)\left(\frac{\sqrt{\lambda}r}{2}\right)^0 I_{\frac{2-2}{2}}(\sqrt{\lambda}r) \tag{4}$$

$$= I_0(\sqrt{\lambda}r) \tag{5}$$

There is a unique relationship between the Jacobi Polynomial and the Bessel function of the first kind [15]. For an arbitrary n, the limit of Jacobi polynomial $P_N^{\nu,\beta}$ as $N \to \infty$ is the Bessel function of the first kind of order ν given as ([13],[15])

$$\mathrm{Lim}_{N\to\infty} \frac{P_N^{\nu,\beta}\left(\mathrm{Cos}z\left(\frac{z}{N}\right)\right)}{N^{\nu}} = \frac{J_\alpha(z)}{\left(\frac{z}{2}\right)^{\alpha}} \tag{6}$$

If $\nu = \beta = \frac{n-2}{2}$, the LHS 0f (6) becomes the Gegenbauer polynomials which are orthogonal polynomials that correspond to the spherical functions associted with the Gelfand pairs $(SO(n+1), SO(n))$. On the RHS of (6), the function $\frac{J_{\frac{n-2}{2}}(z)}{\left(\frac{z}{2}\right)^{\frac{n-2}{2}}}$ is the spherical function associated with the pair $(SO(n) \rtimes \mathbb{R}^n, SO(n))$. For $n = 2$,

$$\frac{J_{\frac{n-2}{2}}(z)}{\left(\frac{z}{2}\right)^{\frac{n-2}{2}}} = J_0(z) \tag{7}$$

and (6) becomes

$$\mathrm{Lim}_{N\to\infty} P_N\left(\mathrm{Cos}\left(\frac{z}{N}\right) = J_0(z)\right. \tag{8}$$

Definition [3]. A spherical measure for the Gelfand pair (G,K) is a non zero Radon measure m on G such that

1. m is K-bi-invariant. That is to say, $m(k_1 t k_2) = m(t), t \in G, k \in K$.
2. $f \mapsto m(f) = \int_G f(g)dm(g)$ is an algebra homomorphism $C_c(K \setminus G/K)) \to \mathbb{C}$. That is, $m(f_1 * f_2) = m(f_1)m(f_2)$.

In otherwords, m is a multiplicative linear functional on $C_c(K \setminus G/K))$.

Definition[4]. A function $f: G \to \mathbb{C}$ is said to be positive definite (written as $f \gg 0$) if the following inequality holds

$$\sum_{i,j=1}^{m} \alpha_i \overline{\alpha_k} f(g_i^{-1} g_k) \geq 0 \tag{9}$$

for all subsets $\{g_1, .., g_m\}$ of elements of G and all sequences $\{\alpha_1, \dots, \alpha_m\}$ of complex numbers.
The integrable analogue of (9) is given as

$$\int_G \int_G f(g_i^{-1} g_k)\varphi(g_i)\varphi(g_k)dg_i g_k \geq 0 \tag{10}$$

where φ ranges over $L^1(G)$ or over the space $C_c(G)$ of continuous functions with compact support. If f is a continuous function, (9) is equivalent to (10). Equation (10) can be re-written as

$$\int_{\infty}^{\infty} f(g)(\varphi * \varphi^*)(g)dg \geq 0 \tag{11}$$

where $\varphi^*(g) = \overline{\varphi(g^{-1})}$ and $*$ denotes the convolution operation. Equation(11) is often taken as the basis for defining positive definite distribution. Positive definite functions have the following properties.
(i) $f(e) \geq 0$
(ii) $|f(g)| \leq |f(e)|, \forall g \in G$.

A positive definite spherical function for the Gelfand pair (G, K) is a positive definite function ϕ on G that is a spherical function for the pair (G, K). A radial function on $\mathbb{R}^2$ is the spherical function for the Gelfand pair $(M(2), SO(2))$. This function is the Bessel function of order zero [see def. 3.4]. It is positive definite and bounded [12]. The next theorem found in ([10]) clearly establishes the relationship between the positive definite spherical function for (G, K) and the unitary representation of G. It will be used in the proof of a type of Bochner's theorem for positive definite spherical function for the Motion group. Here is the theorem.

Theorem. The relationship between positive definite spherical functions for (G,K) and irreducible unitary representation of G with a K -fixed vector is given as follows
(a) Let ϕ be a positive definite spherical function for (G, K). We write (π, u) for the corresponding unitary representation and cyclic unit vector such that $\phi(g) = \langle u, \pi(g)u \rangle$. Then π is irreducible, $\pi(k)u = u$ for all $k \in K$, and u spans the space $\mathcal{H}_\pi^k$
(b) Let π be an irreducible unitary representation of G such that $\mathcal{H}_\pi^k$ is spanned by a single unit vector u. Then $\phi(g) = \langle u, \pi(g)u \rangle$ is a positive definite spherical function for (G, K), and (π, u) corresponds to ϕ up to unitary equivalence.
Choquet's theorem for a locally convex space is required as well, it is stated without proof below.

Theorem (Choquet's)[14, Theorem. 2.2]: Suppose that X is a metrizable convex subset of a locally convex space E and x_o is an element of X . Then there exists a probability measure μ (or Borel measure) on X which represents x_o and is supported by the extreme points of X .

Let B be the set of all normalized elementary positive definite spherical functions for $M(n)$. B is isomorphic with $\mathbb{R}^+$. Let B be endowed with weak $*$ topology induced from $L^1(G)$. The main result of this work is stated below.

3.9 Theorem: Let ϕ be a positive definite spherical function on the Gelfand pair (G, K), where $G = M(n)$ and $K = SO(n)$. Then there exists a Radon measure μ on B such that

$$\phi(g) = \int_B \psi(g) d\mu(\psi), \forall g \in G \tag{12}$$

Proof: Let the set of all positive definite functions ψ on G that satisfies the inequality $\psi(e) \leq 1$ be denoted by T, where $(0,1) = e$ is the identity element of G. Let Q be the set of all positive definite spherical functions. We show that Q is a compact convex subset of the unit sphere of $L^\infty(G)$ and the non zero extreme points of Q coincide with the set B. Let us prove, first, that Q is a compact subset of the unit sphere S of $L^\infty(G)$ in the weak * topology. For a representation π of G on a Hilbert space, let T be a positive definite spherical function that fulfils the properties of theorem 3.8 above, that is

$$T(g) = \langle \pi(g)\xi, \xi \rangle, \forall g \in G \tag{13}$$

and

$$\pi(k)\xi = \alpha(k)\xi \ \ \forall k \in K. \tag{14}$$

We want to show that Q is a closed subset of T. To this end, let us consider a sequence or net $\{\phi_i\}$ in Q, this sequence converges in the weak $*$ topology to an element $\phi \in T$. Next, we show that $\phi(kgk') = \alpha(kk')$ for all $k, k' \in K, g \in G$. To this end, let $k, k' \in K$ and let $f \in L^1(G)$. Then

$$\int_G f(g)\{\phi(kgk') - \alpha(kk')\phi(g)\}dg = \int_G f(g)\{\phi(kgk') - \phi_i(kgk')\}dg + \int_G f(g)\{\phi_i(kgk') - \phi(kgk')\}dg$$
$$= \lambda(k'^{-1}) \int_G \left(f(k^{-1}gk'^{-1})\{\phi(g) - \phi_i(g)\}dg + \alpha(kk') \int_G f(g)\{\phi_i(g) - \phi(g)\}dg \right.$$

where λ is the modular function on G. The RHS converges to zero and therefore, the LHS which is independent of i must be equal to zero for all $f \in L^1(G)$. Next we prove that the set of non-zero extreme points of Q coincide with B . To this end, we show that $\psi_1 \in T, \psi_2 \in Q, \psi_2 - \psi_1 \in T$ imply that $\psi_1 \in Q$. Let π be a corresponding representation of ψ_2 as seen in theorem 3.8. Thus $\psi_2(g) = \langle \pi(g)\xi, \xi \rangle$ for all $g \in G$. Since $\mathcal{H}$ is a Hilbert space, there exists a bounded linear operator M on $\mathcal{H}$ such that M commutes with every $\pi(g)$ and such that $\psi_1(g) = \langle \pi(g)M\xi, \xi \rangle$. Therefore, for $k \in K, k' \in K, g \in G$

$$\begin{aligned} \psi_1(kgk') &= \langle \pi(kgk')M\xi, \xi \rangle \\ &= \langle \pi(g)M\pi(k')\xi, \pi(k^{-1})\xi \rangle \\ &= \alpha(kk')\langle \pi(g)M\xi, \xi \rangle \\ &= \alpha(kk')\psi_1(g) \end{aligned}$$

Therefore, $\psi_1 \in Q$. To conclude the proof, let $\phi \in Q$. Applying Choquet's theorem[thrm 3.9], there exists a Radon measure μ on Q that concentrate on $B(\cong \mathbb{R}^+)$such that

$$\phi = \int_B \psi d\mu(\psi) \tag{15}$$

(15) is a weak Petit integral. For every $f \in L^1(G)$

$$\int_B f(g)\phi(g)dg = \int_B \left\{ \int_G f(g)\psi(g)dg \right\} d\mu(\psi) \tag{16}$$

Applying Fubini's theorem to the RHS of (16), we have

$$\begin{aligned} \int_G f(g)\phi(g)dg &= \int_G f(g) \left\{ \int_B \psi(g)d\mu(\psi) \right\} dg \\ &= \int_G f(g)dg \left\{ \int_B \psi(g)d\mu(\psi) \right\}. \end{aligned}$$

Since $f \in L^1(G)$ is continuous, we have

$$\phi(g) = \int_B \psi(g)d\mu(\psi)$$

This completes the proof.

REFERENCES

Bassey, U. N. and Edeke, U. E. Convolution equation and operators on the Euclidean motion group. JNSPS, Vol.6, issue 4, sept., 2024

Bassey, U. N. and Edeke, U. E., The Radial Part of an Invariant Differential Operator on the Euclidean Motion Group. Journal of the Nigerian Mathematical Society. Vol. 43, issue 3, pp 237-251, 2024.

Barker, W.H.; Positive Definite Distributions on Unimodular Lie Groups. Duke Mathematical Journal. Vol.43, No.1, 71-79,1976.

Bernard, J. M; An Integral Representation for A Class of Positive Definite Functions. Math. Ann. 183, 287-289. 1969.

Chirikjian, G.S., Kyatkin, A.B.; Engineering Applications of NonCommutative Harmonic Analysis. CRC Press, New York. 2001.

Dieudonne, Jean, Gelfand Pairs and Spherical Functions. Internat. J. Math. and Math. Sci. Vol. 2, No. 2 $153-162,1979$

Dijk, Gerrit Van; Introduction to Harmonic Analysis and Generalized Gelfand Pairs. Walter De Gruyter, Berlin. 2009.

Edeke, U.E. and Abuchu, J. A note on the Adjoint representation of SU (2). JP. J. Geo. Topo. 26, No. 2, 139-147. 2021. DOI: http://dx.doi.org/10.17654/GT02602013 9

Edeke, U. E. and Udo, N. E. On Gegenbauer polynomial. Universal Journal of Mathematics and Mathematical Sciences 14(1), 1-7, 2021. http://dx.doi.org/10.17654/UM014010001

Sugiura, M.; Unitary Representation and Harmonic Analysis: An Introduction. North- Holland, New York.1990.

Vilenkin, N. Ja., Klimyk, A.U.; Special Functions and the Theory of Group Representations. Kluwer Academic Publishers. 1991.

Walter, N.; Notes on Semi Direct Product Group. Columbia University, New York. 2013

Wolfgang, C., Frank, F.; Strictly Positive Definite Functions on Generalized Motion Group. Institute of Biomathematics and Biometry, Germany. 2005

Yassine El, M.; On Choquet's Theorem. University of California.

Rocio, D. M., Ines P.; Mehler-Heine Formular: A generalization in the Context of Spherical Functions.arXiv:1807.03904v1[math.RT] 10Jul 2018