DOI: http://dx.doi.org/10.4314/gjmas.v12i1.3

GLOBAL JOURNAL OF MATHEMATICAL SCIENCES VOL. 11, NO. 1&2, 2012: 21-26

 ISSN 1596-6208
www.globaljournalseries.com, Email: info@globaljournalseries.com

ON LU FACTORIZATION ALGORITHM WITH MULTIPLIERS

O. E, NTEKIM, I. M. ESUABANA AND U. E. EDEKE

ABSTRACT

Various algorithm such as Doolittle, Crouts and Choleskyǿs have been proposed to factor a square matrix into a product of L and U matrices, that is, to find L and U such that A = LU; where L and U are lower and upper triangular matrices respectively. These methods are derived by writing the general forms of L and U and the unknown elements of L and U are then formed by equating the corresponding entries in A and LU in a systematic way. This approach for computing L and U for larger values of n will involve many sum of products and will result in n^2 equations for a matrix of order n. In this paper, we propose a straightforward method based on multipliers derived from modification of Gaussion elimination algorithm.

KEY WORDS: Lower and Upper Triangular Matrices, Multipliers.

INTRODUCTION

Let A be a square matrix of order n. An LU factorization or decomposition is a decomposition of the form:

$$A = LU \quad \ldots \tag{1}$$

Where L and U are upper and lower triangular matrices (of the same size) respectively (Horn and Johnson, 1985; Kreyszig, 1993; Morris, 1983; Conte, 1965).

The LU factorization is not unique if one only requires that L be lower triangular and U be upper triangular. It is unique if we assign fixed values to the diagonal elements of either L or U (Conte, 1965; Sastry, 1989; Olayi, 2000; Atkinson, 1993).

LU decomposition is used for solving system of linear equations, calculating matrix determinants and inverse.

THEOREM 1 (EXISTENCE AND UNIQUENESS).

The matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \ldots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{bmatrix} \tag{2}$$

admits an LU factorization if and only if all its principal minors are non singular, that is, if

$$a_{11} \neq 0 \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0 \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0 \ldots \quad |A| \neq 0 \tag{3}$$

(Conte, 1965; Sastry, 1989; Olayi, 2000).

LU DECOMPOSITION ALGORITHMS

We now outline the various procedures or methods that have hitherto been used to factor a square matrix A into a product of L and U matrices. We assume in all the methods that no interchanges will be necessary. The methods we are going to examine involve writing the general forms of L and U and the unknown elements of L and U are then found by equating corresponding entries in A and LU in a systematic way.

O. E, Ntekim, Department of Mathematics/Statistics and Computer Science, University of Calabar, Calaba, Nigeria
I. M. Esuabana, Department of Mathematics/Statistics, Cross River University of Technology, Calabar, Nigeria
U. E. Edeke, Department of Mathematics/Statistics and Computer Science, University of Calabar, Calabar, Nigeria

DOOLITTLE ALGORITHM

In this algorithm, the lower triangular matrix has all diagonal elements equal to 1, whereas the upper triangular matrix U is of the general form. Thus, the elements of the matrices L = (l_{ij}) [with main diagonal1, é , 1] and U = (u_{ij}) in this method are computed from (Schied, 1988):

$$\left.\begin{array}{ll} u_{ij} = a_{1j} & j = 1, 2, é\ n \\ l_{i1} = \dfrac{a_{i1}}{U_{11}}, & i = 2, é\ é\ n \\ u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik}u_{kj} & j = i,.\ é\ n \\ l_{ij} = \dfrac{a_{ij} - \sum_{k=1}^{i-1} l_{ik}u_{ki}}{u_{ii}} & i = j + l, é\ ..n \end{array}\right\} \quad (4)$$

CROUT'S ALGORITHM

In Croutõs algorithm, the matrix U has all diagonal elements equal to 1, whereas L has the general diagonal. Hence, the elements of the matrices

L= (l_{ij}) and U = (u_{ij}) [with main diagonal 1,é , 1] are computed from:

$$\left.\begin{array}{ll} l_{i1} = a_{i1} & i = 1, 2, é\ n \\ u_{1j} = \dfrac{a_{1j}}{l_{11}} & j = 2, é\ n \\ l_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik}u_{kj} & i = j,.\ é\ n \\ u_{ij} = a_{ij} - \dfrac{\sum_{k=1}^{i-1} l_{ik}u_{kj}}{l_{ii}} & j = i+1, é\ n \end{array}\right\} \quad (5)$$

CHOLESKY'S ALGORITHM

For a symmetric positive definite matrix A($A=A^T$, $x^TAx>0\ \forall\ x \neq 0$). We can choose $U = L^T$, thus $u_{ij} = l_{ji}$ and (4) are simplified to (Kreyszig, 1993)

$$\left.\begin{array}{ll} l_{11} = \sqrt{a_{11}} & \\ l_{ii} = \sqrt{a_{11} - \sum_{k=1}^{i-1} l_{ik}^2} & i = 2, é\ .n \\ l_{i1} = \dfrac{a_{i1}}{l_{11}} & \\ l_{ij} = \dfrac{1}{l_{jj}}\left(a_{ij} - \sum_{k=1}^{j-1} l_{ik}l_{kj}\right) & i = j+1, é\ é\ n \end{array}\right\} \quad (6)$$

FACTORIZATION WITH MULTIPIERS

Given an nxn matrix,

$$A = a_{ij} = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \_ & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{bmatrix} \quad \text{é é é é é ..} \quad (7)$$

We want to factor A into the form, A = LU

$$\text{With} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{12} \cdots\cdots & u_{1n} \\ u_{21} & u_{22} & u_{22} \cdots\cdots & u_{2n} \\ 0 & 0 & u_{32} \cdots\cdots & u_{3n} \\ 0 & 0 & 0 \cdots\cdots & u_{nn} \end{bmatrix} \text{é é é é é é ..} \quad (8)$$

$$\text{And } L = \begin{bmatrix} 1 & 0 & 0 \cdots\cdots & 0 \\ l_{21} & 1 & 0 \cdots\cdots & 0 \\ l_{31} & l_{32} & 1.. & 0 \\ l_{n1} & l_{n2} & l_{n3} \cdots\cdots & 1 \end{bmatrix} \text{é é é é é é é} \quad (9)$$

Recall the Gaussian elimination algorithm that for a matrix of order n, the elimination is performed in (n-1) steps, K=1,2é ..n-1. In step K, the elements $a_{ij}^{(k)}$ with i,j>k are transformed according to (Dahlquist and Bjorck; 1974):

$$m_{ik} = \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} \quad \text{..........................} \quad (10)$$

$$a_{ij}^{(k+1)} = a_{ij}^{(k)} - m_{ik} a_{kj}^{(k)} \quad \text{é é é é é ....} \quad (11)$$

i=k+1, k+2,é .n; j=i, i+1,é é n

Where m_{ik} is called the multiplier.

It has been shown by Dahlquist & Bjorck (1974), Scheid(1988) and Matthews(1987) that the elements in L are the multipliers and the matrix U the final triangular matrix obtained by Gaussian elimination.

Hence, we can say that:
$m_{ik} = l_{ik}$
(10) and (11) can now be written as:

$$l_{ik} = \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} \text{é é é é é é é é é é é é é} \quad (12)$$

$$a_{ij}^{(k+1)} = a_{ij}^{(k)} - l_{ik}\, a_{kj}^{(k)} \quad \text{é é é é é é é é é} \quad (13)$$

Also, observe that after triangularisation, (7) will take the form:

$$U = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} \cdots\cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} \cdots\cdots & a_{2n}^{(2)} \\ 0 & 0 & a_{33}^{(3)} .. & a_{3n}^{(3)} \\ 0 & 0 & 0 \cdots\cdots & a_{nn}^{(n)} \end{bmatrix} \quad \text{é é é é .} \quad (14)$$

So, we can let A = a_{ij} in (7) equals $a_{ij}^{(1)}$,

That is, let A = $a_{ij} = a_{ij}^{(1)}$ é é é é é (15)

Comparing (8) with (14), we can say that,

$a_{ij}^{(1)} = u_{ij}$, j = 1 to n é é é é é (16)

we already know that,

l_{ii} =1, i =1 to n é é é é é é (17)

Instead of writing i = k+1, k+2,é .n; j=i, i+1,é n; we can write:
i=2 to n for (12), since for k=1, this transformation begins from row 2 and i, j= 2 to n for (13) since for k=1, it begins from row 2 column 2.
Also comparing (8) with (14) , we can say that:
$a_{ij}^{(i)} = u_{ij}$, i = 2,é .n é é é é é é (18)

Combining (15), (16), (17), (12), (13) and (18) we now write an algorithm for factoring A into LU:

Let A = $a_{ij} = a_{ij}^{(1)}$
$a_{1j}^{(1)} = u_{ij}$, j=1to n
l_{ii} = 1, i = 1 to n
For k=1,2 to n-1
$l_{ik} = \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}}$ i>k, i = 2 to n
$a_{ij}^{(k+1)} = a_{ij}^{(k)} - l_{ik}a_{kj}^{(k)}$ i,j>k, i,j = 2 to n:
$a_{ij}^{(i)} = u_{ij}$ i, j = 2,é n
U = (u_{ij}) 1Òi, j Òn and L = (L_{ij}) 1Òi, j Òn

APPLICATION (Stroud, 1996)
We want to decompose

A = $\begin{bmatrix} 3 & 2 & -1 \\ 2 & -1 & 2 \\ 1 & -3 & -4 \end{bmatrix}$ into A = LU,

Which we know the result to be:

L = $\begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & \frac{11}{7} & 1 \end{bmatrix}$, U = $\begin{bmatrix} 3 & 2 & -1 \\ 0 & -\frac{7}{3} & \frac{8}{3} \\ 0 & 0 & -\frac{55}{7} \end{bmatrix}$

METHOD 1: USING MULTIPLIERS
$a_{11}^{(1)} = 3$, $a_{12}^{(1)} = 2$, $a_{13}^{(1)} = -1$, $a_{21}^{(1)} = 2$, $a_{22}^{(1)} = -1$, $a_{23}^{(1)} = 2$

$a_{31}^{(1)} = 1$, $a_{32}^{(1)} = -3$, $a_{33}^{(1)} = -4$.

$a_{1j}^{(1)} = u_{1j}$, j = 1 to n ⇒

$a_{11}^{(1)} = u_{11} = 3$, $a_{12}^{(1)} = u_{12} = 2$, $a_{13}^{(1)} = u_{13} = -1$

$l_{11} = 1$, i =1 to n ⇒

$l_{11} = l_{22} = l_{33} = 1$

For k = 1 to n-1, we have:

K= 1, i = 2, ⇒ l_{21} =2/3
K= 1, i = 3, ⇒ l_{31} =1/3
K= 1, i = 2, j=2 ⇒ $a_{22}^{(2)}$ =-7/3
K= 1, i = 2, j=3 ⇒ $a_{23}^{(2)}$ =8/3
K= 1, i = 3, j=2 ⇒ $a_{32}^{(2)}$ =-11/3
K= 1, i = 3 j=3, ⇒ $a_{33}^{(2)}$ =-11/3
K= 2, i = 3, ⇒ l_{32} =11/7
K= 2, i = 3 j=3, ⇒ $a_{33}^{(3)}$ =-55/7

Thus,
$a_{22}^{(2)} = U_{22} = -7/3$, $a_{23}^{(2)} = U_{23} = 8/3$, $a_{22}^{(3)} = U_{33} = -55/7$,

$$\therefore \; L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & \frac{11}{7} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 3 & 2 & -1 \\ 0 & -\frac{7}{3} & \frac{8}{3} \\ 0 & 0 & -\frac{55}{7} \end{bmatrix}$$

METHOD 2: USING DOOLITTLE ALGORITHM
For the purpose of our comparison, we shall use Doolittle algorithm.
We already know that, Doolittle algorithm(4) is obtained by writing the general forms of L and U, where L has all the diagonal elements equal to , whereas the upper triangular matrix U is of the general form and the unknown elements of L and U are then found by equating corresponding entries in A and LU in a systematic way. Thus, for:

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & -1 & 2 \\ 1 & 3 & -4 \end{bmatrix}$$

Let $l_{11} =$ $l_{22} =$ $l_{33} =$ 1

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{12} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$LU = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ l_{21} U_{11} & l_{21} U_{12} + U_{22} & l_{21} U_{13} + U_{23} \\ l_{31} U_{11} & l_{31} U_{12} + l_{32} U_{22} & l_{31} U_{13} + l_{32} U_{23} + U_{33} \end{bmatrix}$$

But A = LU $\Rightarrow$ $U_{11} = 3$, $U_{12} = 2$, $U_{13} = -1$

$l_{21} u_{11} = 2$, $\Rightarrow 3 l_{21} = 2$, $\Rightarrow l_{21} = 2/3$

$l_{31} u_{11} = 1$ $\Rightarrow 3\, l_{31} = 1$, $\Rightarrow$ $l_{31} = 1/3$

$l_{21} u_{12} + u_{22} = -1$ $\Rightarrow 4/3 + u_{22} = -1$

$\Rightarrow u_{22} = -1 - 4/3 = -7/3$

$l_{21}\, u_{13} + u_{23} = 2$, $\Rightarrow -2/3 + U_{23} = 2$

$\Rightarrow u_{23} = 2 + 2/3 = 8/3$

$l_{31}\, u_{12} + l_{32}\, u_{22} = -3$, $\Rightarrow 1/3^{(2)} + l_{32}\left(-\frac{7}{3}\right) = 3$
$\Rightarrow l_{32} = 11/7$

$l_{31}\, U_{13} + l_{32}\, U_{23} + U_{33} = -4$

$1/3(-1) + 11/7(8/3) + U_{33} = -4$

$-1/3 + 88/21 + U_{33} = -4$

$U_{33} = -4$ ï $81/21 = -165/21 = -55/7$

$$\text{Thus } A = LU = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & \frac{11}{7} & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & -1 \\ 0 & -\frac{7}{3} & \frac{8}{3} \\ 0 & 0 & -\frac{55}{7} \end{bmatrix}$$

CONCLUSION

We have modified the Gaussian elimination algorithm and have developed a straightforward algorithm based on multipliers for factoring an n x n matrix A into the form A = LU, where L are the multipliers with ls on the diagonal and U is the upper triangular matrix. We have also observed that our proposed algorithm does not involve many sums of products as compared to the Doolittle algorithm.

REFERENCES

Atkinson, K. E., 1993. Elementary Numerical Analysis (2nded). John Wiley & Sons Inc, Canada.

Conte, S. D., 1965. Elementary Numerical Analysis. McGraw – Hill Inc, New York.

Dahlquist, G. and Bjorck, A., 1974. Numerical Methods(Translated by Anderson, N). Prentice – Hall Ine, New Jersey.

Horn, R. A; and Johnson, C. R., 1985. Matrix Analysis. Cambridge University Press, London.

Kreyszig, E. (1993). Advanced Engineering Mathematics (7th ed). John Wiley & Sons Inc, Singapore.

Matthews, J. H. (1987). Numerical Methods for Computer Science, Engineering & Mathematics. Prentice Hall Inc, New Jersey.

Morris, J. L., 1983. Computational Methods In Elementary Numerical Analysis. John Wiley & sons Ltd, Britain.

Olayi, G. A., 2000. Introductory Numerical Methods. Ahmadu Bello University Press, Zaria.

Sastry, S. S., 1989. Introductory Methods of Numerical Analysis (2nded). Printice – Hall of India Private Limited, New Dehli.

Scheid, F. J., 1988. Schaum's Outline Series of Theory and Problems of Numerical Analysis (2nded). Mccraw Hill Companies Inc, USA.

Stroud, K. A., 1996. Further Engineering Mathematics (2nded). Macmillan Press Ltd, London.