GLOBAL JOURNAL OF MATHEMATICAL SCIENCES VOL10, NO.1 & 2, 2011: 27-35
COPYRIGHT© BACHUDO SCIENCE CO. LTD PRINTED IN NIGERIA. ISSN 1596-6208
www.globaljournalseries.com; Email: info@globaljournalseries.com

A HYBRID ITERATIVE SCHEME FOR OPTIMAL CONTROL PROBLEMS GOVERNED BY SOME FREDHOLM INTEGRAL EQUATIONS

AKBAR H. BORZABADI AND MOJTABA AZIZSEFAT

(Received 18, April 2011; Revision Accepted 12, July 2011)

ABSTRACT

This paper presents an iterative approach based on hybrid of perturbation and parametrization methods for obtaining approximate solutions of optimal control problems governed by some Fredholm integral equations. By some numerical examples, it is emphasized that this scheme is very effective and it produces approximate solutions with high precision. Convergence of the given iterative scheme is also discussed.

KEY WORDS: Optimal control problem; Fredholm integral equation; Perturbation method; Parametrization; Approximation.

INTRODUCTION

Trace of numerical approaches in optimal control problems can be found in many applications and academic literatures. One can see in (Schmidt, 2006), an overview of numerical methods for solving optimal control problems described by ODE and integral equations. Despite rapid progress in providing numerical methods for solving optimal control problems, these methods are not much developed for optimal control of some classes of integral equations as Fredholm integral equations. An analysis of finite-element Galerkin discretizations for a class of constrained optimal control problems governed by Fredholm integral and integro-differential equations is given in (Brunner and Yan, 2005). The excellent work of Rubiček (Rubiček, 1998) is one of the best references on the theoretical view of these problems where analytical discussions about existence and uniqueness of the optimal control solution governed by Fredholm integral equations can be found in it.

The current work intends to combine the method of parametrization, (Mehne and Borzabadi, 2006; Teo, et al. 1991; Teo, et al. 1999), and perturbation method, (Biazar and Ghazvini, 2008; Ghasemi, et al., 2007), both are successful methods for solving some classes of optimal control problems and differential-integral equations, respectively, to provide a numerical scheme for approximate optimal control of systems governed by a class of Fredholm integral equations described by the following minimization problem:

$$Minimize \;\; J(x,u) = \int_0^T f_\circ(t, x(t), u(t))dt, \qquad (1)$$

$$subject\ to:$$

$$x(t) = y(t) + \int_0^T k(s,t,u(s))x(s)ds, \;\; a.e.\ on\ [0,T], \qquad (2)$$

where $f_\circ \in C([0,T]\times\mathrm{R}\times\mathrm{R})$ and $k \in C([0,T]\times[0,T]\times\mathrm{R})$. Thereafter, without loss of generality we suppose $T = 1$.

Required perturbation scheme

Recently, the idea of coupling the traditional perturbation method and homotopy in topology (He, 2000), i.e. homotopy perturbation method (HPM), which was proposed first by J. Huan He in 1998 (He, 1998), has been extended and improved by many scientist and engineers as an applicable tool for obtaining approximate solution in a wide range of problems in applied mathematics. Because this method continuously deforms the difficult problem under study into a simple problem which is easier to solve (see e.g. (Saadatmandia, et al., 2009)). Especially, this method has been applied for solving an extensive class of integral equations and we address only (Biazar and Ghazvini, 2008; Dehghan and Shakeri, 2008; Ghasemi, et al., 2007; Javidi and Golbabai, 2009; Saberi-Nadjafi and Ghorbani, 2009).

In this section, we review to apply this method for solving Fredholm integral equations as follows:

$$x(t) = y(t) + \lambda\int_a^b \varphi(s,t,x(s))ds, t \in [a,b], \qquad (3)$$

where λ is a real number and $\varphi \in C([a,b]\times[a,b]\times\mathrm{R})$. For this purpose, first a homotopy as

$$(1-p)(X(t) - x_0(t)) + p(X(t) - y(t) - \lambda\int_a^b \varphi(s,t,x(s))ds) = 0,\ p \in [0,1]. \qquad (4)$$

may be considered where it is assumed that the solution of (4) has the form

Akbar H. Borzabadi, Department of Applied Mathematics, Damghan University, Damghan, Iran.
Mojtaba Azizsefat, Department of Applied Mathematics, Damghan University, Damghan, Iran.

$$X(t)=X_0(t)+pX_1(t)+p^2X_2(t)+\cdots, \tag{5}$$

and $X_i(t), i=1,2,\cdots$ are unknown functions which must be determined. The initial approximation $X_0(t)$ or $x_0(t)$ can be freely chosen, so considering

$$X_0(t)=x_0(t)=y(t). \tag{6}$$

and with substituting (5) into (4), it can be concluded that

$$\begin{cases} p^0: X_0(t)-x_0(t)=0, \\ p^1: X_1(t)-x_0(t)-y(t)-\lambda\int_a^b \varphi(s,t,X_0(s))ds=0, \\ p^2: X_2(t)-\lambda\int_a^b \varphi(s,t,X_1(s))ds=0, \\ \vdots \end{cases} \tag{7}$$

Now the approximate solutions of (4), can be obtained by assuming $p=1$, i.e.

$$x(t)=\lim_{p\to 1}X(t)=\sum_{j=1}^{\infty}X_j(t). \tag{8}$$

Combination of the approaches

In this section, we present a new scheme based on the homotopy perturbation method and method of parametrization. First we consider $\{e_k(t)\}$ as a basis, which is dense in the space of $C([0,1])$. The continuous control function $u(t)$ can be approximated by a finite combination from elements of this basis (Rudin, 1976) and hereby a parametrized form of control function is considered. By a perturbation equation same as (4)

$$(1-p)(X(t)-x_0(t))+p(X(t)-y(t)-\int_0^T \kappa(s,t,u(s))X(s)ds)=0, t\in[0,T], p\in[0,1], \tag{9}$$

our scheme in the k th iteration considers the control parametrized form as

$$u(s)=\sum_{j=0}^{k}c_je_j(s) \tag{10}$$

and a power series such as (5), where $X_i(t,c_0,c_1,\cdots,c_k), i=1,2,\cdots,$ are unknown functions which must be determined. The initial approximations to the solutions $X_0(t)$ are taken to be $y(t)$ as (6). Thus the coefficients of p with the same power lead to

$$\begin{cases} p^0: X_0-y(t)=0, \\ p^1: X_1(t,c_0,c_1,\cdots,c_k)-\int_0^t \kappa(s,t,u(s))X_0(s,c_0,c_1,\cdots,c_k)ds=0, \\ p^2: X_2(t,c_0,c_1,\cdots,c_k)-\int_0^t \kappa(s,t,u(s))X_1(s,c_0,c_1,\cdots,c_k)ds=0 \\ \vdots \end{cases} \tag{11}$$

The approximate solutions of (2), which are dependent on the parameters $c_j, j=0,1,\cdots,k$, can be obtained by setting $p=1$ as follows:

$$x(t,c_0,c_1,\cdots,c_k)=\lim_{p\to 1}X=X_0+X_1+X_2+\cdots. \tag{12}$$

Substituting the trajectory and control (10) and (12) in (1), the solution of an optimization problem as

$$\min_{(c_0,c_1,\cdots,c_k)} J_k(c_0,c_1,\cdots,c_k)=\int_0^T f_\circ(t,x(t,c_0,c_1,\cdots,c_k),\sum_{j=0}^{k}c_je_j(s))dt, \tag{13}$$

may give rise to approximate trajectory and control. It is natural to wonder when iterations in the given procedure can be determined. Assuming J_k^* as optimal value of (13) in the k th iteration, a stopping criteria may be considered as follows:

$$\frac{|J^*_{k+1} - J^*_k|}{|J^*_k|} < \varepsilon, \tag{14}$$

for a prescribed small positive number ε that should be chosen according to the desired accuracy or if the number of iterations exceeds a predetermined number. Note that, there is only one difficulty in the process of applying approach which is to obtain the unknown functions in (11) may be difficult, specially for large n and we consider it in the process of the approach. The above results has been summarized in an algorithm. This algorithm is presented in two stages, initialization step and main steps.

Initialization step:
Choose $\varepsilon > 0$ for the accuracy desired and a dense basis, $\{e_m(t)\}$, for the space $C([0,1])$ and parameter p in the interval $[0,1]$. Set $n = k = 1$, $X_0 = y(t)$ and go to the main steps.

Main steps:
Step 1. Set $u(s)$ by (10), and go to Step 2.
Step 2. Compute $X_n(t, c_0, c_1, \cdots, c_k)$ by (11) if it is possible and go to Step 3, otherwise, $k = k+1$ and go step 1.
Step 3. Compute $(c_0, \cdots, c_k) = arg\min J_k$ in (13) by (12) and go to Step 4.
Step 4. If the stopping criteria (21) holds, stop; Otherwise, $n = n+1$ and go to step 2.

Convergency of the approach
Let Q be the subset of product space $C^\infty([0,1]) \times C^\infty([0,1])$ containing all pairs $(x(\cdot), u(\cdot))$ that satisfy the integral equation (2). Also let Q_k be the subset of Q consisting of all pairs $(x(\cdot), u_k(\cdot))$ where $x(\cdot)$ and $u_k(\cdot)$ are a parameterized control function and trajectory function, respectively obtained in the end of kth iteration of the given algorithm. Let's consider the problem in the Step 3 which is minimization of J on Q_k with $\{c_j\}_{j=0}^k$ as unknowns. This is obviously an optimization problem in $k+1$ dimensional space

$$\{(c_0, c_1, \cdots, c_k) \in \mathbb{R}^{k+1} : a_0 = u_k(0) = u_0, \sum_{j=0}^{k} c_j = u_k(1) = u_1\},$$

and $J(x, u_k)$ may be considered as a function $J(c_0, c_1, \cdots, c_k)$.

Lemma 1 *If* $\alpha_k = \inf_{Q_k} J$ *for* $k = 1, 2, \cdots$, *then* $\{\alpha_k\}_{k=1}^\infty$ *is a convergent sequence.*

Proof: By definition Q_k we have

$$Q_1 \subset Q_2 \subset \cdots \subset Q_k \subset Q_{k+1} \subset \cdots \subset Q,$$

and therefore

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{k+1} \geq \cdots \geq \alpha,$$

Now it can be concluded that $\{\alpha_k\}$ is convergent because it is a nondecreasing and bounded from below sequence. W

Theorem 1 *If* $\lim_{k\to\infty} \alpha_k = \alpha$ *then* $\alpha = \inf_Q J$.

Proof: By Lemma 1, let $\{\alpha_k\}$ be convergent to namely $\hat{\alpha} \geq \alpha$. By contradiction if $\hat{\alpha} > \alpha$, then $\varepsilon = \frac{\hat{\alpha} - \alpha}{2} > 0$. By the properties of infimum, (Rudin, 1976), there exists $(x(\cdot), u(\cdot))$, such that

$$J(x(\cdot), u(\cdot)) < \alpha + \varepsilon = \frac{\hat{\alpha}}{2} + \frac{\alpha}{2}. \tag{15}$$

From the continuity of J, there is a $\delta > 0$ where

$$|J(v(\cdot), w(\cdot)) - J(x(\cdot), u(\cdot))| < \varepsilon, \tag{16}$$

whenever

$$\|(v(\cdot), w(\cdot)) - (x(\cdot), u(\cdot))\|_\infty < \delta. \tag{17}$$

Here $\|\cdot\|_\infty$ is a norm on the vector space $C^\infty([0,1]) \times C^\infty([0,1])$ which can be defined as follows:

$$\|(v(\cdot), w(\cdot))\|_\infty = \|v(\cdot)\|_\infty + \|w(\cdot)\|_\infty,$$

and one can easily check the properties of the norm for it. On the other hand, the set of all polynomial pairs are dense in $C^\infty([0,1]) \times C^\infty([0,1])$, so there is a pair of polynomials $\xi(t)$ and $\upsilon_k(t)$ of the degree at most k such that

$$\|(\xi(\cdot),\upsilon_k(\cdot))-(x(\cdot),u(\cdot))\|_\infty<\frac{\delta}{3}. \qquad (18)$$

Whereas the pair $(\xi(\cdot),\upsilon_k(\cdot))$ does not satisfy

$$(\xi(0),\upsilon_k(0))=(x_0,u_0),(\xi(1),\upsilon_k(1))=(x_1,u_1),$$

we have to define another polynomials

$$v(t)=\xi(t)+(x_0-\xi(0))(1-t)+(x_1-\xi(1))t,\ w_k(t)=\upsilon_k(t)+(u_0-\upsilon_k(0))(1-t)+(u_1-\upsilon_k(1))t,$$

such that satisfy $(v(0),w_k(0))=(x_0,u_0)$ and $(v(1),w_k(1))=(x_1,u_1)$, so $(v,w_k)\in Q_k$.

From (18) for $t=0,1$ we have

$$\|(\xi(0),\upsilon_k(0))-(x_0,u_0)\|_\infty<\frac{\delta}{3},\ \|(\xi(1),\upsilon_k(1))-(x_1,u_1)\|_\infty<\frac{\delta}{3}.$$

Now for $t\in[0,1]$ by definition $v(\cdot)$ and $w_k(\cdot)$ we have

$$\|(v(\cdot),w_k(\cdot))-(x(\cdot),u(\cdot))\|_\infty\le\|(\xi(t),\upsilon_k(t))-(x(t),u(t))\|_\infty$$

$$\|(\xi(0),\upsilon_k(0))-(x_0,u_0)\|_\infty(1-t)+\|(\xi(1),\upsilon_k(1))-(x_1,u_1)\|_\infty t$$

$$<\frac{\delta}{3}+\frac{\delta}{3}+\frac{\delta}{3}=\delta.$$

Therefore

$$\|(v(\cdot),w_k(\cdot))-(x(\cdot),u(\cdot))\|_\infty<\delta,$$

and (16)-(17) imply that

$$|J(v(\cdot),w_k(\cdot))-J(x(\cdot),u(\cdot))\|_\infty<\varepsilon,$$

and so from (15)

$$J(v(\cdot),w_k(\cdot))<\frac{\hat{\alpha}}{2}-\frac{\alpha}{2}+J(x(\cdot),u(\cdot))<\hat{\alpha},$$

a contradiction concludes with $(v(\cdot),w_k(\cdot))\in Q_k$, so $\hat{\alpha}=\alpha$. W

Numerical examples

In this section, some numerical examples are given to show the efficiency of the proposed algorithm. In all examples, $\varepsilon=10^{-5}$ and monomial functions $\{t^k\}$ have been considered as dense basis of $C([0,1])$.

Example 1. Consider the following optimal control problem which is minimization of the functional

$$J(x,u)=\int_0^1(x(t)-t^2)^2+(u(t)-t^2)^2\,dt,$$

governed by Fredholm integral equation

$$x(t)=\frac{2}{3}t^2-\frac{1}{6}+\int_0^1(t^2+su(s))x(s)ds.$$

The exact optimal trajectory and control functions are $x^*(t)=t^2$ and $u^*(t)=t^2$, respectively. Results of applying the proposed algorithm in the previous section have been shown in Table 1. Also the obtained approximate optimal control and trajectory which has been compared to the exact ones can be seen in Fig.1.

Table.1. Numerical results in Example 1.

k	n	J_k^*
1	6	0.0074
2	6	4.7656×10^{-4}
3	6	3.6412×10^{-5}

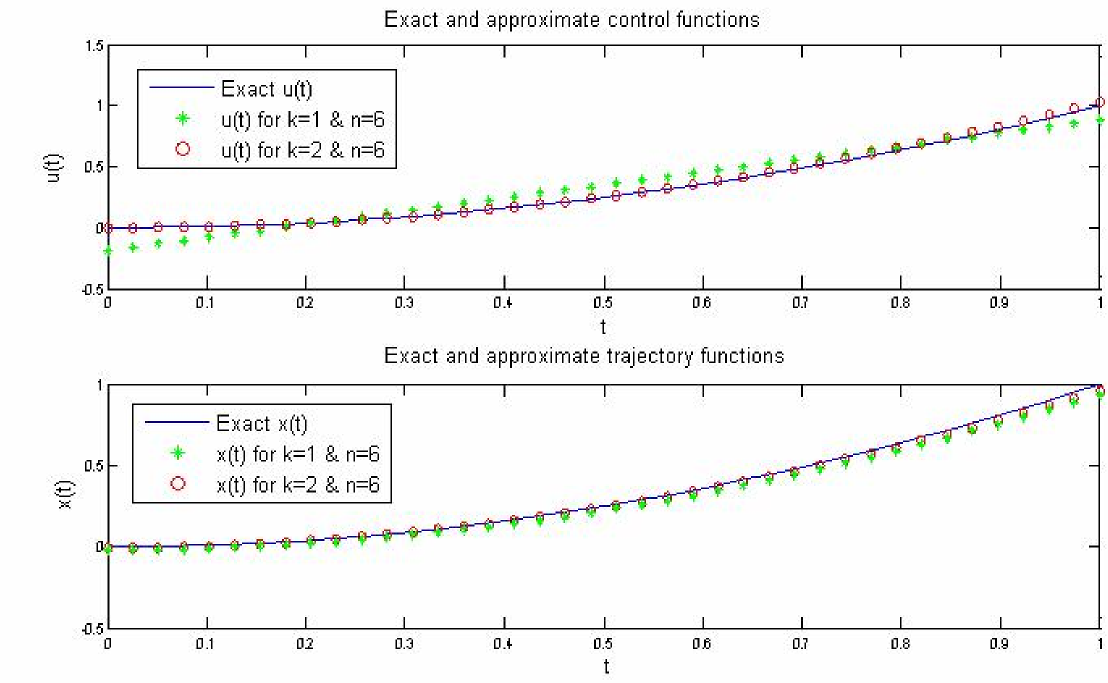

Figure 1: The control and state functions in Example 1.

Example 2. Consider the following optimal control problem

$$Minimize \;\; J(x,u)=\int_0^1 (x(t)-t)^2+(u(t)-t)^2\,dt,$$

subject to :

$$x(t)=-\frac{1}{4}t^2+\frac{5}{6}t-\frac{1}{4}+\int_0^1 (s^3t+st^2+s)u(s)x(s)ds.$$

The exact optimal trajectory and control functions are $x^*(t)=t$ and $u^*(t)=t$, respectively. Results of applying the given algorithm are presented in Table 2. Also, comparison of the exact and approximate optimal control and trajectory may be seen in Fig.2.

Table.2. Numerical results in Example 2.

k	n	J_k^*
1	3	0.0109
2	3	5.6505×10^{-4}
3	3	2.1254×10^{-6}

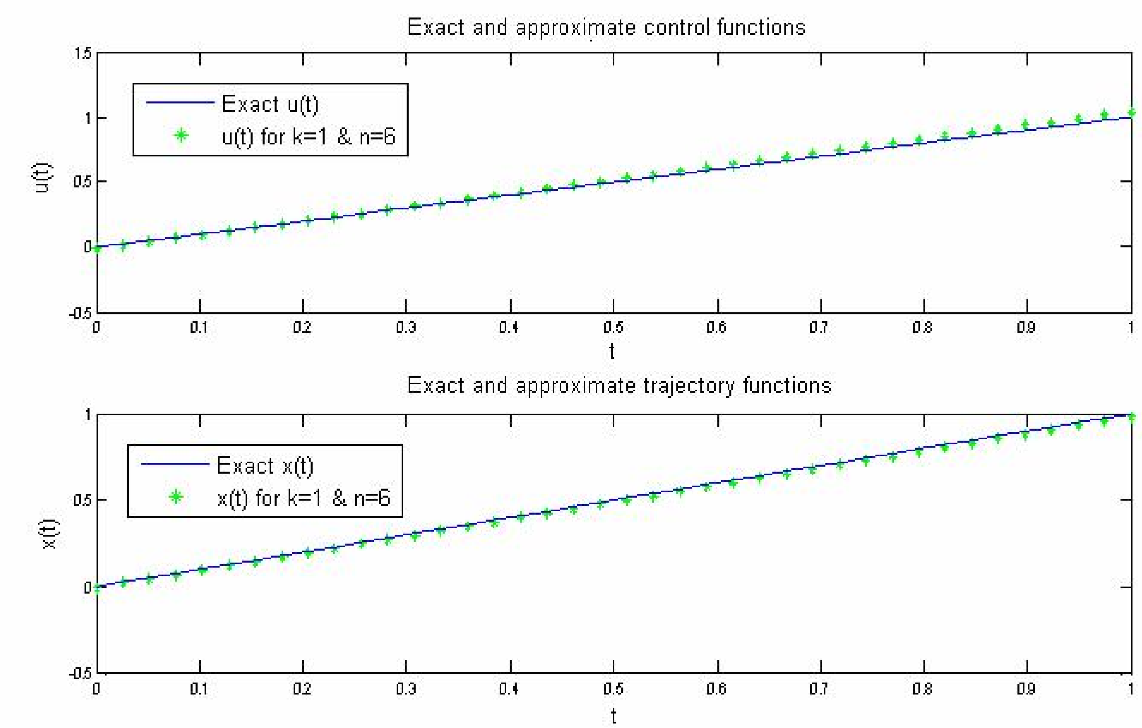

Figure 2: The control and state functions in Example 2.

Example 3. In this example an optimal control problem with different objective function and kernel respect to previous examples is considered as follows

$$Minimize\ \ J(x,u)=\int_0^1 (x(t)-u(t))^2\,dt,$$

$$subject\ to:$$

$$x(t)=e^t-t^2+\int_0^1 st^2e^{-s}u(s)x(s)ds$$

where $x^*(t)=u^*(t)=e^t$ are the exact optimal trajectory and control functions, respectively. The results of applying the given algorithm is presented in Table 3. Also, one can observe the approximate optimal trajectory and control functions which is obtained in some iterations of the given algorithm and compared to the exact solutions in Fig.3.

Table.3. Numerical results in Example 3.

k	n	J_k^*
1	3	0.0049
2	3	2.7836×10^{-5}
3	3	2.3422×10^{-5}

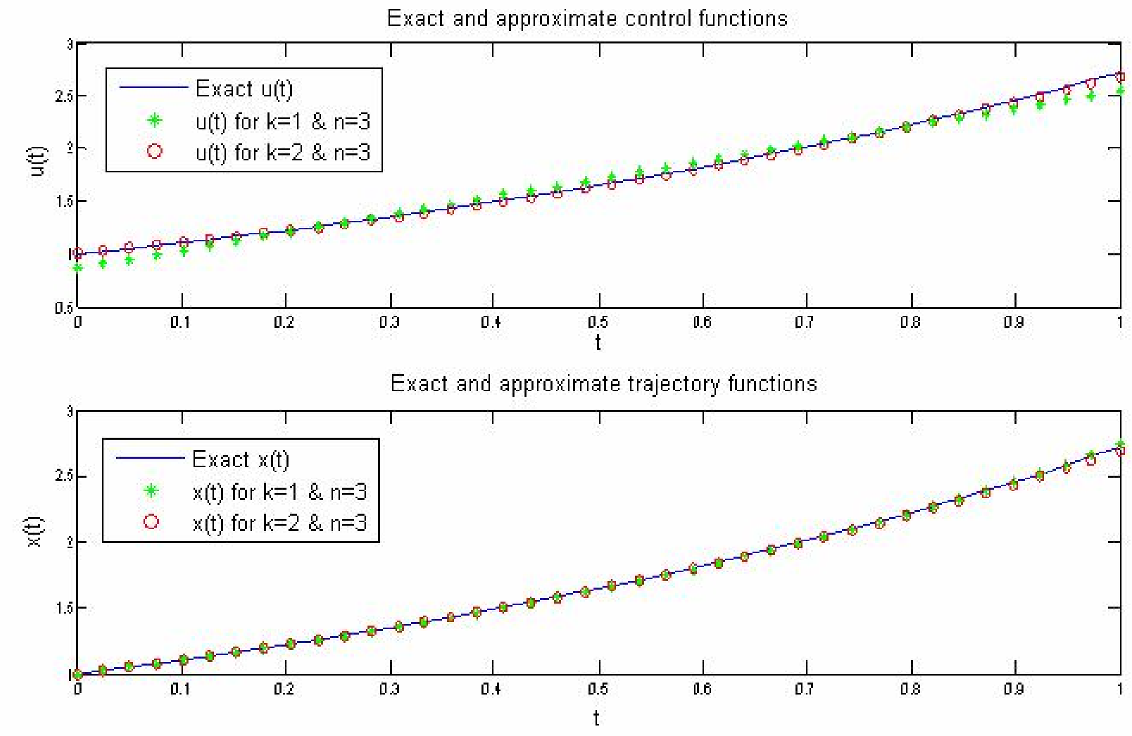

Figure 3: The control and state functions in Example 3.

Example 4. In this example the following optimal control problem is considered

$$Minimize \;\; J(x,u) = \int_0^1 (x(t) - \cos t)^2 (u(t) - t)^2 \, dt,$$

subject to:

$$x(t) = \cos t - t^2(2\cos 1 - \sin 1) + \int_0^1 t^2 u^2(s) x(s) ds.$$

The exact optimal trajectory and control functions are $x^*(t) = \cos t$ and $u^*(t) = t$, respectively. The results of applying the algorithm have been shown in Table 4. The comparison of the exact and approximate optimal control and trajectory may be seen in Fig.4.

Table.4. Numerical results in Example 4.

k	n	J_k^*
1	3	2.3370×10^{-23}
2	3	2.1750×10^{-23}

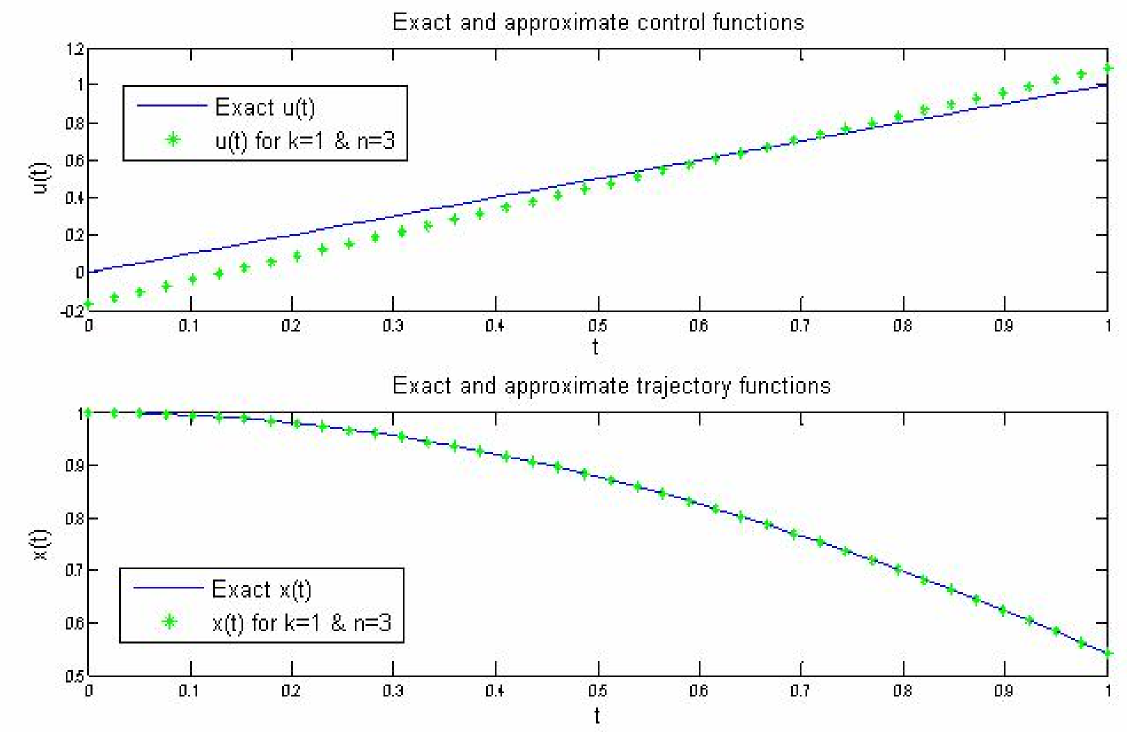

Figure 4: The control and state functions in Example 4.

CONCLUSION

In this article, the perturbation method and parametrization approach are combined for solving optimal control problems governed by a class of Fredholm integral equations. The proposed procedure is very simple and effective and some numerical examples show that the given scheme can produce the approximate solutions with high precision.

REFERENCES

Biazar, J. and Ghazvini, H., 2008. Numerical solution for special non-linear Fredholm integral equation by HPM. Applied Mathematics and Computation 195: 681–687.

Brunner, H. and Yan, N., 2005. Finite element methods for optimal control problems governed by integral equations and integro-differential equations. Numer. Math. 101: 1-27.

Dehghan, M. and Shakeri, F., 2008. Solution of an integro-differential equation arising in oscillating magnetic fileds using He's homotopy perturbation method. Progress In Electromagnetics Research 78: 361–376.

Ghasemi, M., Tavassoli Kajani, M. and Babolian, E., 2007. Numerical solutions of the nonlinear Volterra–Fredholm integral equations by using homotopy perturbation method. Applied Mathematics and Computation 188: 446–449.

He, J.-H., 1998. Approximate solution of nonlinear differential equations with convolution product nonlinearities. Comput. Methods Appl. Mech. Eng. 167: 69–73.

He, J.-H., 2000. A new perturbation technique which is also valid for large parameters. J. Sound Vibration 229: 1257–1263.

Javidi, M. and Golbabai, A., 2009. Modified homotopy perturbation method for solving non-linear Fredholm integral equations. Chaos, Solitons and Fractals 40: 1408–1412.

Mehne, H. H. and Borzabadi, A. H., 2006. A numerical method for solving optimal control problems using state parametrization. J. of Numerical Algorithms 42 (2): 165-169.

Rudin, W., 1976. Principles of Mathematical Analysis. 3rd edn. McGraw-Hill, New York, 342.

Saadatmandia, A., Dehghan, M. and Eftekharia, A., 2009. Application of He's homotopy perturbation method for non-linear system of second-order boundary value problems. Nonlinear Analysis: Real World Applications 10: 1912–1922.

Saberi-Nadjafi, J. and Ghorbani, A., 2009. He's homotopy perturbation method: An effective tool for solving nonlinear integral and integro-differential equations. Computers and Mathematics with Applications 58: 2379--2390.

Schmidt, W. H., 2006. Numerical Methods for Optimal Control Problems with ODE or Integral Equations. Lecture Notes in Computer Science, 3743, Springer Berlin / Heidelberg.

Tahmasbi, A. and Fard, O. S., 2008. Numerical Solution of Nonlinear Volterra Integral Equations of the Second Kind by Power Series. Journal of Information and Computing Science 3 (1): 57-61.

Teo, K. L., Goh, C. J. and Wong, K. H., 1991. A Unified Computational Approach to Optimal Control Problems. Longman Scientific and Technical, Essex.

Teo, K. L., Jennings, L. S., Lee, H. W. J. and Rehbock, V., 1999. Control parametrization enhancing technique for constrained optimal control problems. J. Austral. Math. Soc. B 40: 314-335.

Rubiček, T., 1998. Optimal control of nonlinear Fredholm integral equations. Journal of Optimization and its Applications 19: 707-729.