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ON SOME PROPERTIES OF THE ALTERNATING SYLVESTER SERIES AND ALTERNATING ENGEL SERIES REPRESENTATION OF REAL NUMBERS

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ABSTRACT

We investigate some properties connected with the alternating Sylvester series and alternating Engel Series representation for real numbers, in terms of the integer digits involved. In particular, we look at an algorithm that leads to a general alternating series expansion for real numbers in terms of rationals and deduce the alternating Sylvester and alternating Engel series from this general series.

KEY WORDS: alternating Sylvester series, alternating Engel series, rational numbers, alternating series expansion.

INTRODUCTION

The series of Engel and Sylvester (Galambos 1976)for representing real numbers have been studied in some detail. Much less known is the fact that there are alternating series representations of real numbers in terms of rationals corresponding to the above. In 1989 Knopfmacher and Knopfmacher introduced an algorithm according to which any real number may be expressed by a general alternating series of rationals. This algorithm is described below.

Given any real number A, let $a_0 = [A]$, (where $[A]$ is the interger part).

$A_1 = A - a_0 = \{A\}$ ($\{A\}$ the fractional part). Then recursively define

$$a_n = \left[\frac{1}{A_n}\right] \geq 1, \text{ for } n \geq 1, \quad A_n > 0,$$

where $A_{n+1} = \left(\dfrac{1}{a_n} - A_n\right)\left(\dfrac{c_n}{b_n}\right)$

with $a_n > 0$. Herein

$b_i = b_i(a_1, a_2, \cdots, a_i)$, $C_i = C_i(a_1, a_2, \cdots, a_i)$ are positive numbers

(usually integers), chosen so that $A_n \leq 1$ for $n \geq 1$. Note that

$$A_{n+1} \geq 0, \text{ since } a_n \leq \frac{1}{A_n}, \text{ for } A_n > 0.$$

From the algorithm, the following was proved by Knopfmachor and Knopfmacher (1989).

Theorem 1: Every real number A has a unique representation in the form either (i) or (ii) below:

(i) $$A = a_0 + \frac{1}{a_1} - \frac{1}{(a_1+1)a_1}\frac{1}{a_2} + \frac{1}{(a_1+1)a_1(a_2+1)a_2}\frac{1}{a_3} -$$

$$\equiv (a_0, a_1 \cdots, a_n, \cdots)$$

where $a_i \geq 1$ for $i \geq 1$

(ii) $$A = a_0 + \frac{1}{a_1} - \frac{1}{(a_1+1)}\frac{1}{a_2} + \frac{1}{(a_1+1)(a_1+1)}\frac{1}{a_3} -$$

$$\equiv (a_0, a_1 \cdots, a_n, \cdots)$$

where $a_{i+1} \geq a_i$, $a_1 \geq 1$.

From theorem 1, they obtained the following particular cases (iii) and (iv)

(iii) $$A = a_0 + \frac{1}{a} - \frac{1}{a_2} + \frac{1}{a_3} + \cdots$$

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with $a_{i+1} \geq a_i(a_i+1)$ when $b_n = c_n = 1$ for all n and

(iv) $$A = a_0 + \frac{1}{a_1} - \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} - \cdots$$

where $a_{i+1} \geq a_i + 1 \ (i \geq 1)$, when $c_n = a_n$, $b_n = 1$ for all n.

(iii) above is known in literature as the alternating Sylvester series while (iv) is known as the alternating Engel expansion (Kalpazidou and Ganatsiou (1991)). We are interested in studying the properties of these alternating series.

Theorem 2: The alternating Sylvester and alternating Engel series terminate after a finite number of terms if and only if A is rational.

Proof:

Clearly any number represented by a finite expansion is rational. Conversely, since A_i, $i \geq 1$, is rational let $A_i = \frac{p_i}{q_i}$, $(p_i, q_i) = 1$. Now since for either algorithm $a_i = \left[\frac{1}{A_i}\right] > \frac{1}{A_i} - 1$ it follows that $q_i - p_i\ a_i < p_i$.

In the alternating Sylvester Case we now obtain

$$\frac{p_{i+1}}{q_{i+1}} = \frac{q_i - p_i a_i}{a_i q_i}$$

Thus $0 \leq p_{i+1} \leq q_i - p_i a_i < p_i$. Since $\{p_i\}$ is a strictly decreasing sequence of non-negative integers, we must eventually reach a stage at which $p_{n+1} = 0$, where

$$A = a_0 + \frac{1}{a_1} - \frac{1}{a_2} + \cdots + \frac{(-1)^{n-1}}{a_n} \tag{1}$$

Similarly the result for the alternating Engel series follows from

$$\frac{p_i+1}{q_i+1} = \frac{q_i - p_i a_i}{q_i}.$$

However Jeffrey Shallit (1991), shows that not all rational numbers have representations that terminate, he gave the example of the rational $\frac{2}{2r+1}$ (r an integer ≥ 2), that is neither finite nor ultimately periodic.

Theorem 3: The alternating Sylvester series and the alternating Engel series representations of real numbers are unique.

Proof:

Recall that the Engel alternating series for a rational A is

$$A = a_0 + \frac{1}{a_1} - \frac{1}{(a_1+1)} \cdot \frac{1}{a_2} + \frac{1}{(a_1+1)(a_2+1)} \cdot \frac{1}{a_3} - \cdots \tag{2}$$

Here a_0 is an integer and a_i is a positive integer for $i \geq 1$. If $a_{i+1} \geq a_i$, for all i then expansion is essentially unique. Similar proof for Sylvester's series.

Note that for rational numbers with a finite expansion there is a possible ambiguity in the final term. This ambiguity is eliminated by the conventions below, introduced by Kalpazidou and Ganatsiou (1991).

Convention 1: Replace the finite sequence $((a_0, a_1, \cdots, a_{n-1}+1))$ by the sequence $((a_0, a_1, \cdots, a_{n-2}, a_{n-1}+1))$ in the case $a_n = 1$.

Convention 2: Replace the finite sequence $(a_0, a_1, \cdots, a_n)$ by the sequence $(a_0, a_1, \cdots, a_{n-2}, a_{n-1}+1)$ in the case $a_n = a_{n-1}$.

Order Property in Alternating Sylvester and Engel Expansions.

In order to be able to compare finite expansions of different lengths in size we introduce the symbol Ω with the property $n < \Omega$, for any $n \in N$. We can represent finite sequences by infinite sequences as follows: for every

$A = ((a_0, a_1, \cdots, a_n))$ let $a_j = \Omega$ for

$j > n$ and hence $A = ((a_0, a_1, \cdots, a_n, \Omega, \cdots))$.

The same apply in the case

$$A = (a_0, a_1, \cdots, a_n)$$

Theorem 4: (Order property). Let $A((a_0\,,\,a_1,\cdots)) \neq B = ((b_0\,,\,b_1,\cdots,b_1,\cdots))$ or $A = (a_0\,,\,a_1\cdots) \neq B = (b_0\,,\,b_1\cdots)$ in both cases, $A < B$ is equivalent to

(i) $a_{2n} < b_{2n}$ or (ii) $a_{2n+1} > b_{2n+1}$

where $i = 2n$ or $i = 2_{n+1}$ is the first index $i \geq 0$ such that $a_i \neq b_i$.

Proof:

Let $A_n^1 = \dfrac{1}{a_n} - \dfrac{1}{(a_n+1)} \cdot \dfrac{1}{a_{n+1}} + \dfrac{1}{(a_n+1)(a_{n+1}+1)} \dfrac{1}{a_{n+2}} - \cdots$

For $A = ((a_0\,,\; a_1,\, a_2\cdots))$ and

$$A_n^1 = \frac{1}{a_n} - \frac{1}{(a_n+1)a_n} \cdot \frac{1}{a_{n+1}} + \frac{1}{(a_n+1)(a_{n+1}+1)a_{n+1}} \cdot \frac{1}{a_{n+2}} - \cdots$$

for $A = (a_0\,,\; a_1,\, a_2\cdots)$. No assumption is made that $A_n^1 = A_n$.

Now suppose (i) holds. If firstly $a_0 < b_0$ then

$A = a_0 + A_1 < a_0 + 1 \leq b_0 \leq b_0 + B_1 = B$ in either case.

Next suppose $a_{2n} < b_{2n}\,,\; n > 0,$ in the alternating Sylvester Case, since

$a_{i+1} \geq a_i(a_1+1),\; i \geq 1$, we have

$$\begin{aligned}
A_{2n}^1 &= \frac{1}{a_{2n}} - \frac{1}{a_{2n+1}} + \frac{1}{a_{2n+2}} - \cdots \\
&\geq \frac{1}{a_{2n}}\left(1 - \frac{1}{a_{2n}+1}\right) + \frac{1}{a_{2n+2}}\left(1 - \frac{1}{a_{2n+2}} + 1\right) + \cdots \\
&> \frac{1}{a_{2n}}\left(1 - \frac{1}{a_{2n}} + 1\right) \\
&\quad = \frac{1}{a_{2n}+1}
\end{aligned}$$

Since $a_i > 1 \text{ for } i > 1,$ and by observing convention 1. Furthermore

$$\begin{aligned}
A_{2n}^1 &= \frac{1}{a_{2n}} - \frac{1}{a_{2n+1}} + \frac{1}{a_{2n+2}} - \cdots \\
&\leq \frac{1}{a_{2n}} - \frac{1}{a_{2n+1}}\left(1 - \frac{1}{a_{2n+1}} + 1\right) - \frac{1}{a_{2n+3}}\left(1 - \frac{1}{a_{2n+3}} + 1\right) - \cdots \\
&\leq \frac{1}{a_{2n}}
\end{aligned}$$

Thus $A_{2n}^1 > \dfrac{1}{a_{2n}+1} \geq \dfrac{1}{b_{2n}} \geq B_{2n}^1$

It follows from $A = a_0 + \dfrac{1}{a_1} - \dfrac{1}{a_2} + \cdots - A_{2n}^1$,

$B = a_0 + \dfrac{1}{a_1} - \dfrac{1}{a_2} + \cdots - B_{2n}^1$,

that $A < B$.

In the alternating Engel Case, since

$a_{i+1} \geq a_i + 1,\; i \geq 1$

we have

$$A_{2n}^1 = \frac{1}{a_{2n}} - \frac{1}{a_{2n}a_{n+1}} + \frac{1}{a_{2n}\; a_{2n+}\; a_{2n+2}} - \cdots$$

$$\geq \frac{1}{a_{2n}}\left(1-\frac{1}{a_{2n}+1}\right) + \frac{1}{a_{2n}\ a_{2n+1}\ a_{2n+2}}\left(1-\frac{1}{a_{2n+2}}+1\right)+\cdots$$
$$> \frac{1}{a_{2n}}\left(1-\frac{1}{a_{2n}+1}\right)$$
$$= \frac{1}{a_{2n}+1}$$

as in the Sylvester Case Also

$$A_{2n}^{1} = \frac{1}{a_{2n}} - \frac{1}{a_{2n}a_{2n+1}} + \frac{1}{a_{2n}a_{2n+1}\ a_{2n+2}} - \cdots$$
$$\leq \frac{1}{a_{2n}} - \frac{1}{a_{2n}\ a_{2n+1}}\left(1-\frac{1}{a_{2n+1}}+1\right) - \cdots$$
$$\leq \frac{1}{a_{2n}}$$

Thus again

$$A_{2n}^{1} > \frac{1}{a_{2n}+1} \geq \frac{1}{b_{2n}} > B_{2n}^{1}$$

and the result $A < B$ follows from

$$A = a_0 + \frac{1}{a_1} + \frac{1}{a_1\,a_2} + \cdots - \frac{1}{a_1 a_2 \cdots a_{2n-1}}\ A_{2n}^{1}$$
$$B = a_0 + \frac{1}{a_1} - \frac{1}{a_1\,a_2} + \cdots - \frac{1}{a_1 a_2 \cdots a_{2n-1}}\ B_{2n}^{1}$$

Note if $b_{2n} = \Omega$ then $B_{2n}^{1} = 0$ and the result remains valid.

The result is proved in a similar way if (ii) holds.

CONCLUSION

We have considered the alternating Sylvester and alternating Engel series expansions for rational numbers. Like the positive series, the corresponding alternating series terminates and is unique for every rational number $A = \frac{a}{b}$, $b \neq 0$. Also using a simple method we have shown that both the alternating Sylvester series and the alternating Engel series are well ordered.

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