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STABILITY RESULTS OF A MATHEMATICAL MODEL FOR THE CONTROL OF HIV/AIDS WITH THE USE OF MALE AND FEMALE CONDOMS IN HETEROSEXUAL POPULATIONS

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ABSTRACT

A compartmentalized deterministic mathematical model for the dynamics of HIV/AIDS under the use of male and female condoms has been formulated and studied qualitatively. Disease-free equilibria of the sub-models have been found to be locally and asymptotically stable. Stability results revealed threshold values for the proportions of susceptible and infected subpopulations that must use condom in order to achieve control, and possibly, eradication of HIV/AIDS in heterosexual populations. Condom use rate for the susceptible subpopulations has been found to be bounded above by the population's birth rate, while that of the infected subpopulations is bounded below by a given threshold.

KEYWORDS: Locally and asymptotically stable, disease-free equilibrium, HIVAIDS control

1 INTRODUCTION

Research has revealed a great deal of valuable medical, scientific and public health information about the human immunodeficiency virus (HIV), the causative agent for acquired immunodeficiency syndrome (AIDS), (Avert, 2008). HIV is spread by sexual contact with infected persons, by sharing hypodermic instruments (primarily for drug injection) with infected persons or through transfusion of infected blood (Avert, 2008). Babies born to HIV infected women may become infected before or during birth or breast feeding (CDC, 1999). Although the HIV prevalence is much lower in Nigeria than in other African countries such as South Africa and Zambia, the size of Nigeria's population (around 138 million) meant that by the end of 2007, there were an estimated 2,600,000 people infected with HIV, which is rather on the high side (UNAIDS, 2008).

Condoms are classified as medical devices and are regulated by the food and drug administrations across various countries of the world. There are many different types and brands of condoms available – however, only latex and polyurethane condoms provide highly effective barrier to HIV (CDC, 1993). For condom to provide maximum protection, they must be used consistently and correctly (CDC, 1988).

Once a sexually active HIV infected person is introduced into a heterosexual population, the virus begins to spread in the population due to interactions between susceptible and infective members of the population. The deployment of male and female condoms is aimed at reducing the chances of spreading and/or acquiring sexually transmitted diseases (STDs) including HIV. Condoms do not offer full protection against HIV and other STDs, but can significantly reduce the chances of transmitting and/or acquiring same (CDC, 1998).

About 80 per cent of HIV infections in Nigeria are transmitted through heterosexual contacts (Avert, 2008). The total number of condoms provided by international donors has been relatively low. Between 2000 and 2005, the average number of condoms distributed in Nigeria by donors was 5.9 per man, per year (UNFPA, 2005). A study in 2002 found that 75 percent of health service facilities that had been visited did not have any condoms or contraceptive supplies (Human Rights Watch, 2004). The number of female condoms sold in Nigeria has significantly increased, which indicates a greater awareness of sexual health issues. In 2003 only 25,000 female condoms had been sold, which increased to 375,000 in 2006 (UNFPA, 2007). One major advantage of the female condom is that it does not rely upon the willingness of the man to use a condom himself (Family Health International, 2007).

A number of mathematical models on HIV/AIDS have been proposed and studied in order to gain better insight into the dynamics of the disease both in vivo and invitro (see Kimbir (2005); Renton, Whitaker and Riddlesdell (1998); Rauner, Brailsford and Flessa (2005); Korobeinikov (2004); Henry and Frederic (2000); Greenhalgh, Doyle and Lewis (2001); and Corbett, Moghadas, and Gumel (2003)).

Our main objective in this paper is to investigate the existence of the conditions for eradication of HIV in heterosexual populations under the use of male and female condoms. This is significant because the efforts channeled in the provision, promotion and counseling on the use of condom may amount to a waste if not optimally conducted. Two things are critical here; the subpopulations that should use condom, and the proportion of same subpopulations that must use the condom for effective results

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2 THE MODEL EQUATIONS

Consider a heterosexual population with condom deployed in the various subpopulations. Assume that a proportion each of both infected and susceptible males and females use the condom (condoms are not 100 per cent efficacious). AIDS is an additional cause of dead for the infected persons.

Compartmentalizing the population into susceptible males not using the condom (S_m), susceptible females not using the condom (S_f), susceptible males using the condom (U_m), susceptible females using the condom (U_f), infected males not using the condom (I_m), infected females not using the condom (I_f), infected males using the condom (W_m), and infected females using the condom (W_f), and taking into consideration the epidemiological flow diagram in fig.2.1, which we put forward for the purpose of this research, we obtain the model equations given by equations (2.1) – (2.8).

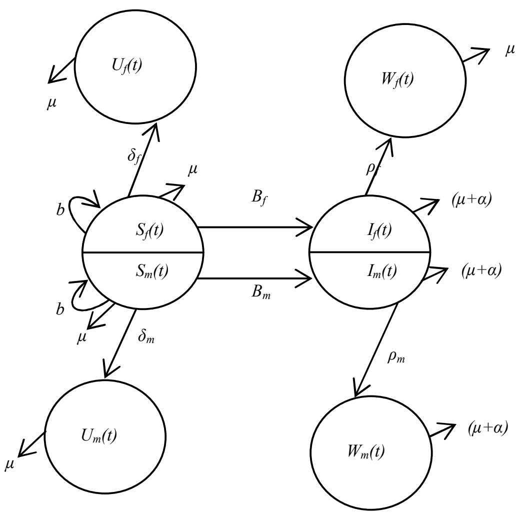

Figure 2.1: Epidemiological flow-diagram for HIV/AIDS dynamics incorporating the use of male and female condoms

We derive the model equations by considering the assumptions and the epidemiological flow diagram (fig.2.1). The incidence rates B_m and B_f are given as in Hseih (1996).

$$\frac{dS_m}{dt} = b(S_m + I_m) - B_m S_m - (\mu + \delta_m)S_m \tag{2.1}$$

$$\frac{dS_f}{dt} = b(S_f + I_f) - B_f S_f - (\mu + \delta_f)S_f \tag{2.2}$$

$$\frac{dI_m}{dt} = B_m S_m - (\mu + \alpha + \rho_m)I_m \tag{2.3}$$

$$\frac{dI_f}{dt} = B_f S_f - (\mu + \alpha + \rho_f)I_f \tag{2.4}$$

$$\frac{dU_m}{dt} = \delta_m S_m - \mu U_m \tag{2.5}$$

$$\frac{dU_f}{dt} = \delta_f S_f - \mu U_f \tag{2.6}$$

$$\frac{dW_m}{dt} = \rho_m I_m - (\mu + \alpha)W_m \tag{2.7}$$

$$\frac{dW_f}{dt} = \rho_f I_f - (\mu + \alpha)W_f \tag{2.8}$$

$$B_m = \frac{c_m(\beta_f I_f + \beta_f^* W_f)}{N_f}, \tag{2.9}$$

$$B_f = \frac{c_f(\beta_m I_m + \beta_m^* W_m)}{N_m}, \tag{2.10}$$

$$N_m = S_m + I_m + U_m + W_m, \tag{2.11}$$

$$N_f = S_f + I_f + U_f + W_f, \tag{2.12}$$

where c_m, c_f, β_m, β_f, β_m^* and β_f^* are the average number of sexual contacts for males, average number of sexual contacts for females, probability of transmission by infected males not using condom, probability of transmission by infected female not using the condom, probability of transmission by infected male using condom, and probability of transmission by infected female using the condom, respectively. Similarly, ρ_m, ρ_f, δ_m, δ_f represent the proportions of infected males, proportion of infected females, proportion of susceptible males and proportion of susceptible females using the condom, respectively.

3.0 MODEL EQUATIONS IN PROPORTIONS

Transforming the model equations into proportions has the advantage of reducing the total number of equations in the model, giving the equations in epidemiologically meaningful forms (for instance, the proportion of infected persons defines the prevalence of infection), (Avert, 2008).

Consider the substitutions below,

$$x_m(t) = \frac{S_m(t)}{N_m(t)},\ x_f(t) = \frac{S_f(t)}{N_f(t)},\ y_m(t) = \frac{I_m(t)}{N_m(t)},\ y_f(t) = \frac{I_f(t)}{N_f(t)},\ u_m(t) = \frac{U_m(t)}{N_m(t)},\ u_f(t) = \frac{U_f(t)}{N_f(t)},\ w_m(t) = \frac{W_m(t)}{N_m(t)},\ w_f(t) = \frac{W_f(t)}{N_f(t)}$$

Using these substitutions in equations (2.1) – (2.8) and considering (2.9) to (2.12), we obtain the model equations in proportions as given by equations (3.1) – (3.6)

$$y_m'(t) = c_m\left(\beta_f y_f + \beta_f^* w_f\right)\left(1 - u_m - w_m - y_m\right) - y_m\left[b\left(1 - u_m - w_m\right) + \rho_m + \alpha\left(1 - w_m - y_m\right)\right] \tag{3.1}$$

$$y_f'(t) = c_f\left(\beta_m y_m + \beta_m^* w_m\right)\left(1 - u_f - w_f - y_f\right) - y_f\left[b\left(1 - u_f - w_f\right) + \rho_f + \alpha\left(1 - w_f - y_f\right)\right] \tag{3.2}$$

$$u_m'(t) = \delta_m\left(1 - u_m - w_m - y_m\right) - u_m\left[b\left(1 - u_m - w_m\right) - \alpha\left(y_m + w_m\right)\right] \tag{3.3}$$

$$u_f'(t) = \delta_f\left(1 - u_f - w_f - y_f\right) - u_f\left[b\left(1 - u_f - w_f\right) - \alpha\left(y_f + w_f\right)\right] \tag{3.4}$$

$$w_m'(t) = \rho_m y_m - w_m\left[b\left(1 - u_m - w_m\right) + \alpha\left(1 - y_m - w_m\right)\right] \tag{3.5}$$

$$w_f'(t) = \rho_f y_f - w_f\left[b\left(1 - u_f - w_f\right) + \alpha\left(1 - y_f - w_f\right)\right] \tag{3.6}$$

4 STABILITY ANALYSIS

We apply Hurwitz criterion (see David (1997) and Weisstein (2008)) to study the stability of the disease-free equilibrium (DFE) states of the various sub-models in this paper.

4.1 The sub-model with only infected males using the condom

Here we analyze the stability of the DFE when only infected males use the condom. The sub-model in this case is obtained by setting $u_m = u_f = w_f = 0$ and $\delta_m = \delta_f = \rho_f = 0$ in the general model. We therefore have the system of equations (4.1.1) for the sub-model. The conditions for stability have been stated and proved in the theorem that follows.

$$\begin{aligned} y_m'(t) &= c_m \beta_f y_f (1 - y_m) - y_m\left[b(1 - w_m) + \rho_m + \alpha(1 - w_m - y_m)\right] \\ y_f'(t) &= c_f(\beta_m y_m + \beta_m^* w_m)(1 - y_f) - y_f\left[b + \alpha(1 - y_f)\right] \\ w_m'(t) &= \rho_m y_m - w_m\left[b(1 - w_m) + \alpha(1 - y_m - w_m)\right] \end{aligned} \tag{4.1.1}$$

Theorem 4.1

Given that $\rho_m, c_m, \beta_f, c_f, \beta_m, b, \alpha > 0$. The DFE for the system (4.1.1) is LAS if the inequality

$$\rho_m > \frac{3(b+\alpha)^2 - c_m\beta_f c_f \beta_m}{2(b+\alpha)} \text{ holds.}$$

Proof

The DFE, $E_0 = (0,0,0)$

The Jacobian matrix evaluated at the DFE is given by

$$J_{(E_0)} = \begin{bmatrix} -b-\alpha-\rho_m & c_m\beta_f & 0 \\ c_f\beta_m & -b-\alpha & c_f\beta_m^* \\ \rho_m & 0 & -b-\alpha \end{bmatrix}$$

The characteristic polynomial is given by

$$p(\lambda) = -\lambda^3 - [3(b+\alpha) + \rho_m]\lambda^2 - [3(b+\alpha)^2 + 2\rho_m(b+\alpha) - c_m\beta_f c_f \beta_m]\lambda$$
$$-[(b+\alpha)^3 + \rho_m(b+\alpha)^2 - c_m\beta_f c_f \beta_m(b+\alpha) - \rho_m c_m \beta_f c_f \beta_m^*] = 0$$

For the characteristic polynomial to satisfy Hurwitz criterion (David, 1997), the inequality (4.1.2) must be satisfied

$$\rho_m > \frac{3(b+\alpha)^2 - c_m\beta_f c_f \beta_m}{2(b+\alpha)} \qquad (4.1.2)$$

It can be shown that the right-hand-side of (4.1.2) is positive if the inequality $c_m\beta_f c_f\beta_m < (b+\alpha)^2$ holds.

4.2 The sub-model with only Infected Females using the condom

Here we analyze the stability of the DFE when only infected females use the condom. The sub-model in this case is obtained by setting $u_f = u_m = w_m = 0$ and $\delta_f = \delta_m = \rho_m = 0$ in the general model. We therefore have the system of equations (4.2.1) for the sub-model

$$\begin{aligned} y_m'(t) &= c_m(\beta_f y_f + \beta_f^* w_f)(1-y_m) - y_m[b + \alpha(1-y_m)] \\ y_f'(t) &= c_f\beta_m y_m(1-w_f - y_f) - y_f[b(1-w_f) + \rho_f + \alpha(1-y_f - w_f)] \\ w_f'(t) &= \rho_f y_f - w_f[b(1-w_f) + \alpha(1-y_f - w_f)] \end{aligned} \qquad (4.2.1)$$

Theorem 4.2

Given that $\rho_f, c_m, \beta_f, c_f, \beta_f^*,\ b, \alpha > 0$. If the inequality

$$\rho_f > \frac{3(b+\alpha)^2 - c_m\beta_f c_f \beta_m}{2(b+\alpha)}$$ holds, then the DFE is LAS.

Proof

The DFE, $E_0 = (0,0,0)$

The Jacobian matrix evaluated at the DFE is given by

$$J_{(E_0)} = \begin{bmatrix} -b-\alpha & c_m\beta_f & c_m\beta_f^* \\ c_f\beta_m & -b-\rho_f-\alpha & 0 \\ 0 & \rho_f & -b-\alpha \end{bmatrix}$$

The characteristic polynomial is given by

$$p(\lambda) = -\lambda^3 - [3(b+\alpha) + \rho_f]\lambda^2 - [3(b+\alpha)^2 + 2\rho_f(b+\alpha) - c_m\beta_f c_f\beta_m]\lambda$$
$$-(b+\alpha)^3 - \rho_f[(b+\alpha)^2 - c_f\beta_m c_m\beta_f^*] + (b+\alpha)c_m\beta_f c_f\beta_m = 0$$

For the eigenvalues to be all negative, the inequality (4.2.2) must be satisfied.

$$\rho_f > \frac{3(b+\alpha)^2 - c_m\beta_f c_f \beta_m}{2(b+\alpha)} \qquad (4.2.2)$$

4.3 The sub-model with only susceptible males using condom

Here we analyze the stability of the DFE when only susceptible males use condom. The sub-model in this case is obtained by setting $u_f = w_m = w_f = 0$ and $\delta_f = \rho_f = \rho_m = 0$ in the general model. We therefore have the system of equations (4.3.1) for the sub-model

$$\begin{aligned} y_m'(t) &= c_m\beta_f y_f(1-u_m - y_m) - y_m[b(1-u_m) + \alpha(1-y_m)] \\ y_f'(t) &= c_f\beta_m y_m(1-y_f) - y_f[b + \alpha(1-y_f)] \\ u_m'(t) &= \delta_m(1-u_m - y_m) - u_m[b(1-u_m) - \alpha y_m] \end{aligned} \qquad (4.3.1)$$

Theorem 4.3

Given that $\delta_m, c_m, \beta_f, c_f, \beta_m,\ b, \alpha > 0$. If the inequality

$$\delta_m < \frac{b[(b+\alpha)^2 - c_m\beta_f c_f\beta_m]}{b(b+\alpha) - c_m\beta_f c_f\beta_m}$$ holds, then the DFE is LAS.

Proof

The DFE, $E_0 = (0, 0, \frac{\delta_m}{b})$

The Jacobian matrix evaluated at the DFE is

$$J_{(E_0)}=\begin{bmatrix}-b-\alpha+\delta_m-\lambda & c_m\beta_f\left(\frac{b-\delta_m}{b}\right) & 0\\ c_f\beta_m & -b-\alpha-\lambda & 0\\ \delta_m\left(\frac{\alpha-b}{b}\right) & 0 & -b+\delta_m-\lambda\end{bmatrix}$$

The characteristic polynomial is given by

$$p(\lambda)=-\lambda^3-(3b+2\alpha-2\delta_m)\lambda^2$$
$$-\left[(b-\delta_m)(-2(b+\alpha)+\delta_m)+(b+\alpha)^2-(b+\alpha)\delta_m-c_m\beta_f c_f\beta_m\left(\frac{b-\delta_m}{b}\right)\right]\lambda$$
$$-(b-\delta_m)\left[(b+\alpha)^2-(b+\alpha)\delta_m-c_m\beta_f c_f\beta_m\left(\frac{b-\delta_m}{b}\right)\right]$$

The Hurwitz criterion is satisfied by the characteristic polynomial provided the inequality (4.3.2) holds.

$$\delta_m<\frac{b\left[(b+\alpha)^2-c_m\beta_f c_f\beta_m\right]}{b(b+\alpha)-c_m\beta_f c_f\beta_m} \tag{4.3.2}$$

The RHS of (4.3.2) is positive provided the inequality (4.3.3) holds

$$c_m\beta_f c_f\beta_m<(b+\alpha)^2 \tag{4.3.3}$$

4.4 Sub-model with only susceptible females using condom

Here we analyze the stability of the DFE when only susceptible females use condom. The sub-model in this case is obtained by setting $u_m = w_m = w_f = 0$ and $\delta_m = \rho_m = \rho_f = 0$ in the general model. We therefore have the system of equations (4.4.1) for the sub-model

$$\begin{aligned} y_m'(t)&=c_m\beta_f y_f(1-y_m)-y_m\left[b+\alpha(1-y_m)\right]\\ y_f'(t)&=c_f\beta_m y_m(1-u_f-y_f)-y_f\left[b(1-u_f)+\alpha(1-y_f)\right]\\ u_f'(t)&=\delta_f(1-u_f-y_f)-u_f\left[b(1-u_f)-\alpha y_f\right]\end{aligned} \tag{4.4.1}$$

Theorem 4.4

Given that $\delta_f, c_m, \beta_f, c_f, \beta_m, b, \alpha>0$. If the inequality

$$\delta_f<\frac{b\left[(b+\alpha)^2-c_m\beta_f c_f\beta_m\right]}{b(b+\alpha)-c_m\beta_f c_f\beta_m}$$ holds, then the DFE is LAS.

Proof

The DFE, $E_0=\left(0,0,\frac{\delta_f}{b}\right)$

The Jacobian matrix evaluated at the DFE is

$$J_{(E_0)}=\begin{bmatrix}-b-\alpha & c_m\beta_f & 0\\ c_f\beta_m\left(\frac{b-\delta_f}{b}\right) & -b-\alpha-\delta_f & 0\\ 0 & -\delta_f\left(\frac{b-\alpha}{b}\right) & -b+\delta_f\end{bmatrix}$$

The characteristic polynomial is given by

$$p(\lambda)=-\lambda^3-(3b+2\alpha-2\delta_f)\lambda^2$$
$$-\left[(b-\delta_f)(2(b+\alpha)-\delta_f)+(b+\alpha)^2-(b+\alpha)\delta_f-c_m\beta_f c_f\beta_m\left(\frac{b-\delta_f}{b}\right)\right]\lambda$$
$$-(b-\delta_f)\left[(b+\alpha)^2-(b+\alpha)\delta_f-c_m\beta_f c_f\beta_m\left(\frac{b-\delta_f}{b}\right)\right]$$

The Hurwitz criterion is satisfied by the characteristic polynomial provided the inequality (4.4.2) holds.

$$\delta_f<\frac{b\left[(b+\alpha)^2-c_m\beta_f c_f\beta_m\right]}{b(b+\alpha)-c_m\beta_f c_f\beta_m} \tag{4.4.2}$$

Observe that the right-hand-side of the inequality (4.4.2) is positive if $c_m\beta_f c_f\beta_m<(b+\alpha)^2$

Therefore the DFE is locally and asymptotically stable. Thus the theorem is proved.

4.5 Sub-model with both Susceptible and Infected males using the condom

Here we study the stability of the model in a situation where only males (a proportion each of susceptible and infected males) use the condom.

In this case, $u_f = w_f = 0$ and $\delta_f = \rho_f = 0$, and the sub-model is given by equations (4.5.1).

$$\begin{aligned}
y_m'(t) &= c_m \beta_f y_f (1-u_m-w_m-y_m) - y_m\left[b(1-u_m-w_m)+\rho_m+\alpha(1-w_m-y_m)\right] \\
y_f'(t) &= c_f(\beta_m y_m + \beta_m^* w_m)(1-y_f) - y_f\left[b+\alpha(1-y_f)\right] \\
u_m'(t) &= \delta_m(1-u_m-w_m-y_m) - u_m\left[b(1-u_m-w_m)-\alpha(y_m+w_m)\right] \\
w_m'(t) &= \rho_m y_m - w_m\left[b(1-u_m-w_m)+\alpha(1-y_m-w_m)\right]
\end{aligned} \tag{4.5.1}$$

Theorem 4.5

Given that $b, \alpha, c_m, \beta_f, c_f, \beta_m^*, \rho_m, \delta_m > 0$. If $\delta_m < b$, $\rho_m > 2\delta_m - 3(b+\alpha)$, $c_m\beta_f c_f \beta_m^* < (b+\alpha)^2$, then the DFE is LAS.

Proof

The DFE is obtained as $E_0 = \left(0, 0, \frac{\delta_m}{b}, 0\right)$

The Jacobian matrix evaluated at the DFE is as follows

$$J_{(E_0)} = \begin{bmatrix} -b+\delta_m-\rho_m-\alpha & c_m\beta_f\left(\frac{b-\delta_m}{b}\right) & 0 & 0 \\ c_f\beta_m & -(b+\alpha) & 0 & c_f\beta_m^* \\ -\delta_m\left(\frac{b-\alpha}{b}\right) & 0 & -b+\delta_m & \alpha\frac{\delta_m}{b} \\ \rho_m & 0 & 0 & -(b-\delta_m+\alpha) \end{bmatrix}$$

The characteristic polynomial $p(\lambda)$ is given by

$$p(\lambda) = \begin{vmatrix} -b+\delta_m-\rho_m-\alpha-\lambda & c_m\beta_f\left(\frac{b-\delta_m}{b}\right) & 0 & 0 \\ c_f\beta_m & -b-\alpha-\lambda & 0 & c_f\beta_m^* \\ -\delta_m\left(\frac{b-\alpha}{b}\right) & 0 & -b+\delta_m-\lambda & \alpha\frac{\delta_m}{b} \\ \rho_m & 0 & 0 & -b+\delta_m-\alpha-\lambda \end{vmatrix} = 0$$

The first root of the polynomial $p(\lambda)$ is

$$\lambda_1 = -b+\delta_m$$

$$\mathrm{Re}\,\lambda < 0 \quad iff \quad \delta_m < b$$

The remaining roots of $p(\lambda)$ are the roots of the cubic equation $q(\lambda)$ given by

$$\begin{aligned}
q(\lambda) = & -\lambda^3 - \left[3(b+\alpha) - 2\delta_m + \rho_m\right]\lambda^2 \\
& - \left[(b+\alpha)(b+\alpha-\delta_m+\rho_m) - c_m\beta_f c_f \beta_m^*\left(\tfrac{b-\delta_m}{b}\right) - (b+\alpha-\delta_m)\left((b+\alpha)-\delta_m+\rho_m\right)\right]\lambda \\
& -(b-\delta_m+\alpha)\left[(b+\alpha)(b+\alpha-\delta_m+\rho_m) - c_m\beta_f c_f\beta_m^*\left(\tfrac{b-\delta_m}{b}\right)\right]
\end{aligned}$$

The Hurwitz criterion is satisfied by the characteristic polynomial provided the inequalities (4.5.2) and (4.5.3) hold.

$$\rho_m > 2\delta_m - 3(b+\alpha) \tag{4.5.2}$$

$$c_m\beta_f c_f \beta_m^* < (b+\alpha)^2, \quad \delta_m < b \tag{4.5.3}$$

4.6 Sub-model with both Susceptible and Infected females using condom

Here we study the stability of the model in a situation where only females (a proportion each of susceptible and infected females) use the condom.

In this case, $u_m = w_m = 0$ and $\delta_m = \rho_m = 0$, so that the model becomes

$$\begin{aligned}
y_m'(t) &= c_m(\beta_f y_f + \beta_f^* w_f)(1-y_m) - y_m\left[b+\alpha(1-y_m)\right] \\
y_f'(t) &= c_f\beta_m y_m(1-u_f-w_f-y_f) - y_f\left[b(1-u_f-w_f)+\rho_f+\alpha(1-w_f-y_f)\right] \\
u_f'(t) &= \delta_f(1-u_f-w_f-y_f) - u_f\left[b(1-u_f-w_f)-\alpha(y_f+w_f)\right] \\
w_f'(t) &= \rho_f y_f - w_f\left[b(1-u_f-w_f)+\alpha(1-y_f-w_f)\right]
\end{aligned} \tag{4.6.1}$$

Theorem 4.6

Given that $b,\alpha,c_m,\beta_f,c_f,\beta_f^*,\rho_f,\delta_f>0$. If $\delta_f<b$, $\rho_f>2\delta_f-3(b+\alpha)$, $c_m\beta_f^*c_f\beta_m<(b+\alpha)^2$, then the DFE is LAS.

Proof

The DFE, $E_0=\left(0,0,\frac{\delta f}{b},0\right)$

The Jacobian matrix evaluated at the DFE is as follows

$$J_{(E_0)}=\begin{bmatrix}-b-\alpha & c_m\beta_f & 0 & c_f\beta_f^* \\ c_f\beta_m\left(\frac{b-\delta f}{b}\right) & -b+\delta_f-\rho_f+\alpha & 0 & 0 \\ 0 & -\delta_f\left(\frac{b-\alpha}{b}\right) & -b+\delta_f & \alpha\frac{\delta f}{b} \\ 0 & \rho_f & 0 & -b+\delta_f-\alpha\end{bmatrix}$$

The characteristic polynomial $p(\lambda)$ is given by

$$p(\lambda)=\begin{vmatrix}-b-\alpha-\lambda & c_m\beta_f & 0 & c_m\beta_f^* \\ c_f\beta_m\left(\frac{b-\delta f}{b}\right) & -b+\delta_f-\rho_f+\alpha-\lambda & 0 & 0 \\ 0 & -\delta_f\left(\frac{b-\alpha}{b}\right) & -b+\delta_f-\lambda & \alpha\frac{\delta f}{b} \\ 0 & \rho_f & 0 & -b+\delta_f-\alpha-\lambda\end{vmatrix}=0$$

The first root of the polynomial $p(\lambda)$ is

$$\lambda_1=-b+\delta_f$$

$$\operatorname{Re}\lambda<0 \quad iff \quad \delta_f<b$$

The remaining roots of $p(\lambda)$ are the roots of the cubic equation $q(\lambda)$ given by

$$\begin{aligned}q(\lambda)=&-\lambda^3-\left[3(b+\alpha)-2\delta_f+\rho_f\right]\lambda^2\\&-\left[(b+\alpha)(b+\alpha-\delta_f+\rho_f)-c_m\beta_f c_f\beta_m\left(\tfrac{b-\delta f}{b}\right)-(b+\alpha-\delta_f)\left(2(b+\alpha)-\delta_f+\rho_f\right)\right]\lambda\\&-(b-\delta_f+\alpha)\left[(b+\alpha)(b+\alpha-\delta_f+\rho_f)-c_m\beta_f c_f\beta_m\left(\tfrac{b-\delta f}{b}\right)+\rho_f c_m\beta_f^* c_f\beta_m\left(\tfrac{b-\delta f}{b}\right)\right]\end{aligned}$$

The Hurwitz criterion is satisfied by the characteristic polynomial provided the inequalities (4.6.2) and (4.6.3) hold.

$$\rho_f>2\delta_f-3(b+\alpha) \tag{4.6.2}$$

$$\delta_f<b,\ c_m\beta_f^*c_f\beta_m<(b+\alpha)^2 \tag{4.6.3}$$

4.7 The Effect of suspending the condom

Here we study the dynamics of HIV when no condoms are employed in the entire population.

We set $u_f=w_f=w_m=u_m=0$ and the parameters $\rho_m=\rho_f=\delta_m=\delta_f=0$ to obtain the sub-model as given by the system of equations (4.7.1).

$$\begin{aligned}y_m'(t)&=c_m\beta_f y_f(1-y_m)-y_m\left[b+\alpha(1-y_m)\right]\\y_f'(t)&=c_f\beta_m y_m(1-y_f)-y_f\left[b+\alpha(1-y_f)\right]\end{aligned} \tag{4.7.1}$$

Theorem 4.7

Given that the parameters $c_m,\beta_f,c_f,\beta_m,\ b,\alpha>0$. If the inequality

$c_m\beta_f c_f\beta_m<(b+\alpha)^2$ holds, then the DFE is LAS.

Proof

The DFE is obtained as E_0 = (0, 0)

The Jacobian matrix evaluated at the DFE is *E*

$$J_{(E_0)}=\begin{bmatrix}-b-\alpha & c_f\beta_m \\ c_f\beta_m & -b-\alpha\end{bmatrix}$$

We see that

$$TrJ_{(0,0)}=-2(b+\alpha)<0 \qquad \text{since } b,\alpha>0$$

$$\det J_{(0,0)}=(b+\alpha)^2-c_m\beta_f c_f\beta_m>0,$$

Provided

$$c_m \beta_f c_f \beta_m < (b+\alpha)^2 \qquad (4.7.2)$$

Therefore the DFE of this sub-model is locally and asymptotically stable (LAS)
Thus, the theorem is proved.

DISCUSSION OF RESULTS

The results stated in theorems 4.1 to 4.6 show that eradication is achievable when condom use rates by the susceptible and infected males and females satisfy the specified conditions. The results also reveal that it is sufficient to target condom provisions and promotions at the infected subpopulations and achieve eradication in heterosexual populations. The result in theorem 4.7 agrees with the finding in Kimbir (2005). Therefore, reducing the net transmission rate of HIV either by reducing the average number of sexual contacts or the probability of transmission by infectives is capable of controlling the spread of the virus, possibly, towards eradication.

CONCLUSION

All the sub-models studied in this paper have been found to have stable DFE under different conditions. Stable DFE physically implies that disease eradication is achievable under the specified conditions. Based on the stability results obtained in this paper, rate of condom use by the susceptible subpopulations is bounded above by the population's birth rate, while the rate of condom use by the infected subpopulations is bounded below by the threshold as stated in the various theorems.

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