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ON CERTAIN DIFFERENTIAL EQUATIONS WITH DYNAMICALLY SLOWLY VARYING COEFFICIENTS AND WITH APPLICATION TO DYNAMIC BUCKLING

A. M. ETTE

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ABSTRACT

This investigation is concerned with the solution of a nonlinear dynamical system where the coefficients of the ensuing differential equations are dynamically slowly varying. The formulation contains two small but mathematically independent parameters on which a generalization of Linstedt-Poincare perturbation procedures are executed based on asymptotic expansions of the variables. To the order of the accuracy retained in this work, it is deduced that the dynamic buckling load of the structure investigated is dependent on the first derivative of the load function evaluated at the initial time.

KEY WORDS: Differential equations, Dynamically slowly varying coefficients, Dynamic buckling

1. INTRODUCTION

Nonlinear dynamical systems with slowly varying coefficients were first investigated by Kuzmak (1959) in a study that was limited to formulations that yielded second order differential equations. Later, Luke (1966) extended the investigations to include formulations yielding higher orders differential equations. Much later, similar studies were initiated by Kevorkian (1987), Li (1987) and Amazigo and Ette (1987), among others.

In this paper, we extend the work of Danielson (1969) on the dynamic buckling of an elastic quadratic structure under a step load, to the case where the loading history is dynamically slowly varying over a natural period of vibration of the structure.

2. Formulation

In his investigation, Danielson (1969), considered a two-arm simply-supported column, each of the two arms of length L, with a mass M_1 attached at the meeting point of the two arms of the column and from where a nonlinear spring, with spring constant K_1, is attached, (Fig. 1).

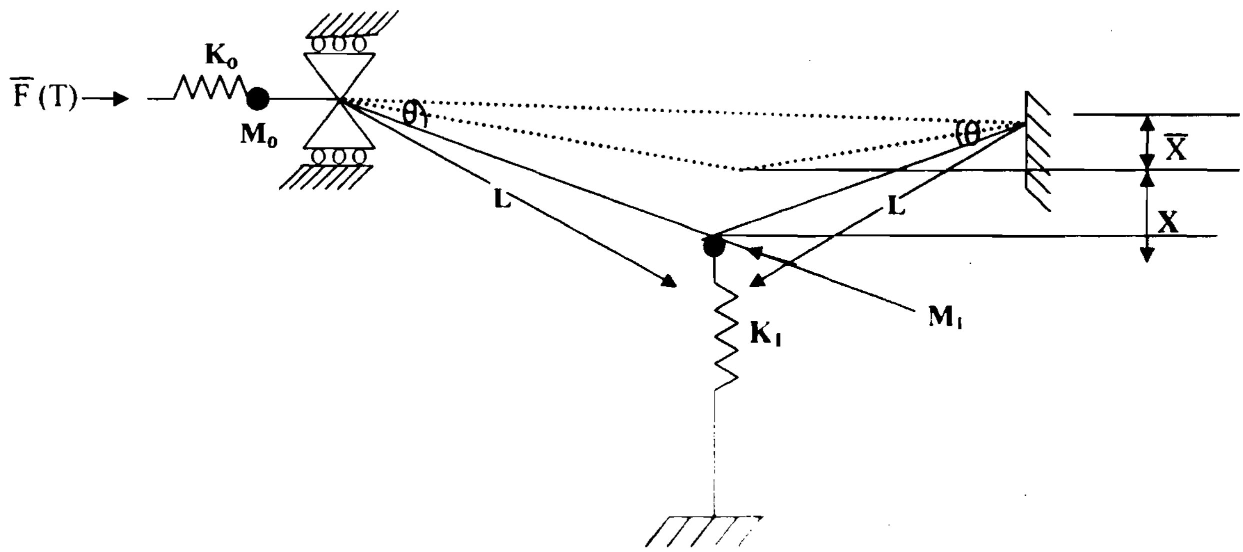

Fig. 1

A simple Quadratic – Elastic Model Structure

A. M. Ette, Department of Mathematics and Computer Science. Federal University of Technology, Owerri. Imo State. Nigeria

The nonlinear spring exerts a force per unit length of $K_1(X - aX^2)$, where $a > 0$ is a constant and $X(T)$ is an additional displacement from the equilibrium position. The entire column is deemed rigid and weightless and carries another mass M_0, and yet, another spring, with spring constant K_0 on the axial direction from where a force $\bar{F}(T)$ is directed at the entire system, with T as the time variable. The role of the mass M_0, and spring with spring constant K_0, is to initiate a pre-buckling motion $X_0(t)$. We let the initial displacement, otherwise called the initial imperfection, be represented by $\bar{X}$. Danielson derived the following coupled equations for the dynamic equilibrium of the structure, which we have here refined with the inclusion of an explicitly time dependent slowly varying load function $\bar{f}(T)$, thus:

$$\frac{1}{\omega_0^2}\frac{d^2\xi_0}{d\hat{T}^2} + \xi_0 - \frac{K_0}{\lambda_c}\xi_1(\xi_1 + 2\bar{\xi}) = \lambda \bar{f}(\hat{T}) \tag{2.1}$$

$$\frac{1}{\omega_1^2}\frac{d^2\xi_1}{d\hat{T}^2} + \xi_1(1-\xi_0) - \alpha\xi_1^2 + \frac{K_0}{\lambda_c}\xi_1(\xi_1 + \bar{\xi})(\xi_1 + 2\bar{\xi}) = \bar{\xi}\xi_0 \tag{2.2}$$

$$\xi_0(0) = \xi_1(0) = \frac{d\xi_0(0)}{d\hat{T}} = \frac{d\xi_1(0)}{d\hat{T}} = 0 \tag{2.3}$$

$$\lambda_c = \frac{K_1}{2},\ \omega_0 = \left(\frac{K_0}{M_0}\right)^{\frac{1}{2}},\ \omega_1 = \left(\frac{K_1}{M_1}\right)^{\frac{1}{2}},\ \alpha = aL^2,\ \bar{\xi} = \frac{\bar{X}}{L},\ \xi_0 = \frac{X_0}{L},\ \xi_1 = \frac{X}{L},\ \bar{f}(\hat{T}) = \frac{\bar{F}(T)}{\bar{F}(0)},\ \hat{T} = T\left(\frac{K_1}{M_1}\right)^{\frac{1}{2}}$$

Here, λ_c, is the classical buckling load, while λ is a nondimensional load parameter that has been nondimensionalized with respect to the classical buckling load λ_c, and thus satisfies the inequality $0 < \lambda < \lambda_c$, while α is the imperfection sensitivity parameter. If we let $\hat{t} = \omega_0\hat{T}$, we get

$$\frac{d^2\xi_0}{d\hat{t}^2} + \xi_0 - \frac{K_0}{\lambda_c}\xi_1(\xi_1 + 2\bar{\xi}) = \lambda\, \mathrm{f}(\delta\hat{t}) \tag{2.4}$$

$$\frac{d^2\xi_1}{d\hat{t}^2} + Q^2\xi_1(1-\xi_0) - Q^2\alpha\xi_1^2 + \frac{K_0 Q^2}{\lambda_c}(\xi_1^3 + 2\bar{\xi}\xi_1^2 + 2\xi_1\bar{\xi}^2) = Q^2\bar{\xi}\xi_0 \tag{2.5}$$

$$\xi_0(0) = \xi_1(0) = \frac{d\xi_0(0)}{d\hat{t}} = \frac{d\xi_1(0)}{d\hat{t}} = 0. \tag{2.6}$$

$$\mathrm{f}(\delta\hat{t}) = \bar{f}\left(\frac{\hat{t}}{\omega_0}\right),\ Q = \left(\frac{\omega_1}{\omega_0}\right),\ 1 < \left(\frac{\omega_1}{\omega_0}\right) < 1$$

We consider $0 < \delta < 1$, $0 < \bar{\xi} < 1$ and note that δ and $\bar{\xi}$ are two small parameters that are not related mathematically. We consider the load function $f(\delta\hat{t})$ to be continuous and slowly varying over a natural period of vibration of the structure. It also possesses right hand derivatives of all orders at $\hat{t} = 0$ and satisfies the following conditions

$$f(0) = 1,\ |\mathrm{f}(\delta\hat{t})| \le 1,\quad \text{for}\quad \hat{t} > 0 \tag{2.7}$$

Except for equation (2.7), $f(\delta\hat{t})$ is strictly arbitrary. Our intension is to solve the equations (2.4)-(2.7), and by so doing obtain the dynamic buckling load λ of the structure. We define the dynamic buckling load λ_D as the largest load parameter for which the solution of equations (2.4)-(2.7) remains bounded for all time $\hat{t} > 0$. According to Budiansky (1966), and Amazigo and Ette (1987), the condition for obtaining λ_D is the maximization

$$\frac{d\lambda}{d\xi_a} = 0,\ \xi_a = \xi_{0a} + \xi_{1a} \tag{2.8}$$

where ξ_{0a} and ξ_{1a} are the maximum values of $\xi_0(\hat{t})$ and $\xi_1(\hat{t})$ respectively, which are the pre-buckling mode and the buckling mode in that order mentioned .Analysis enunciated here is a combination of various techniques developed by many authors , including Wang and Tian (2002a,2002b, 2003),Wei et al (2005) and Batra and Wei (2005), among others. We now initiate an asymptotic solution of the problem with a view to first determining ξ_a, as in (2.8).

3 Asymptotic solution

As noted by Danielson, if $0 < Q < 1$, then we can conveniently neglect the pre-buckling inertia $\frac{d^2\xi_0}{d\hat{t}^2}$, with the result that , from (2.4) and (2.5),we now have

$$\xi_0(\hat{t}) = \lambda f(\delta \hat{t}) + \frac{K_0}{\lambda_C}\xi_1(\xi_1 + 2\bar{\xi}) \tag{3.1}$$

$$\frac{d^2\xi_1}{d\hat{t}^2} + Q^2\xi_1(1-\lambda f) - Q^2\xi_1^2\left(\alpha + \frac{K_0\bar{\xi}}{\lambda_C}\right) = Q^2\bar{\xi}\lambda f \tag{3.2}$$

$$\xi_0(0) = \xi_1(0) = \frac{d\xi_0(0)}{d\hat{t}} = \frac{d\xi_1(0)}{d\hat{t}} = 0 \tag{3.3}$$

where (3.2) is obtained by substituting (3.1) in (2.5) . We now let

$$\tau = \delta \hat{t} \; ; \frac{d\tilde{t}}{d\hat{t}} = \left(1 - \lambda f(\delta \hat{t})\right)^{\frac{1}{2}} = \left(1 - \lambda f(\tau)\right)^{\frac{1}{2}} ; \; t = \tilde{t} + \frac{1}{\delta}\left\{\mu_1(\tau)\bar{\xi} + \mu_2(\tau)\bar{\xi}^2 + \cdots\right\} \tag{3.4a}$$

$$\mu_i(0) = 0, \; i = 1,2,3,\cdots \tag{3.4b}$$

Thus we now have

$$\frac{d\xi_k}{d\hat{t}} = \left\{(1-\lambda f)^{\frac{1}{2}} + \mu_1'\bar{\xi} + \mu_2'\bar{\xi}^2 + \cdots\right\}\xi_{k,t} + \delta\xi_{k,\tau} \tag{3.5}$$

$$\frac{d^2\xi_k}{d\hat{t}^2} = \left[(1-\lambda f) + 2(1-\lambda f)^{\frac{1}{2}}\left\{\mu_1'\bar{\xi} + \mu_2'\bar{\xi}^2 + \cdots\right\} + \left\{\mu_1'\bar{\xi} + \mu_2'\bar{\xi}^2 + \cdots\right\}^2\right]\xi_{k,tt}$$

$$+ 2\delta\left\{(1-\lambda f)^{\frac{1}{2}} + \mu_1'\bar{\xi} + \mu_2'\bar{\xi}^2 + \cdots\right\}\xi_{k,t\tau} + \delta^2\xi_{k,\tau\tau} + \delta\left\{-\frac{\lambda f'}{2(1-\lambda f)^{\frac{1}{2}}} + \mu_1''\bar{\xi} + \mu_2''\bar{\xi}^2 + \cdots\right\}\xi_{k,t} \tag{3.6}$$

where $k = 0,1$, and $\frac{d(\;)}{d\tau} = (\;)'$. Here, a subscript following a comma indicates partial differentiation. We let

$$\xi_1 = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\zeta^{ij}(t,\tau)\bar{\xi}^i\delta^j \tag{3.7}$$

where the $i\,j$, as in ζ^{ij},are superscripts and not powers. We now substitute (3.5) and (3.6) into (3.2) and get the following equations of integral orders of $\bar{\xi}^i\delta^j$:

$$L_1\zeta^{10} \equiv \zeta^{10}_{,tt} + Q^2\zeta^{10} = Q^2 B(\tau), \left(B(\tau) = \frac{\lambda f}{1-\lambda f} \right) \tag{3.8}$$

$$L_1\zeta^{11} = -2(1-\lambda f)^{-\frac{1}{2}}\zeta^{10}_{,t\tau} + \frac{\lambda f'(1-\lambda f)^{-\frac{3}{2}}}{2}\zeta^{10}_{,t} \tag{3.9}$$

$$L_1\zeta^{12} = -2(1-\lambda f)^{-\frac{1}{2}}\zeta^{11}_{,t\tau} + \frac{\lambda f'(1-\lambda f)^{-\frac{3}{2}}}{2}\zeta^{11}_{,t} - \frac{\zeta^{10}_{,\tau\tau}}{1-\lambda f} \tag{3.10}$$

$$L_1\zeta^{20} = \frac{\alpha Q^2(\zeta^{10})^2}{1-\lambda f} - 2(1-\lambda f)^{-\frac{1}{2}}\mu_1'\zeta^{10}_{,tt} \tag{3.11}$$

$$L_1\zeta^{21} = \frac{2\alpha Q^2\zeta^{10}\zeta^{11}}{1-\lambda f} - 2(1-\lambda f)^{-\frac{1}{2}}\mu_1'\zeta^{11}_{,tt} - 2(1-\lambda f)^{-\frac{1}{2}}\zeta^{20}_{,t\tau} + \frac{\lambda f'(1-\lambda f)^{-\frac{3}{2}}}{2}\zeta^{20}_{,t}$$
$$-\frac{\mu_1''\zeta^{10}_{,t}}{1-\lambda f} - \frac{\mu_1'\zeta^{10}_{,t\tau}}{1-\lambda f} \tag{3.12}$$

$$L_1\zeta^{21} = \frac{\alpha\{2\zeta^{10}\zeta^{12} + (\zeta^{11})^2\}}{1-\lambda f} - 2(1-\lambda f)^{-\frac{1}{2}}\mu_1'\zeta^{12}_{,tt} - 2(1-\lambda f)^{-\frac{1}{2}}\zeta^{21}_{,t\tau} + \frac{\lambda f'(1-\lambda f)^{-\frac{3}{2}}}{2}\zeta^{21}_{,t}$$
$$-\frac{\mu_1''\zeta^{11}_{,t}}{1-\lambda f} - \frac{\mu_1'\zeta^{11}_{,t\tau}}{1-\lambda f} - \frac{\zeta^{20}_{,\tau\tau}}{1-\lambda f} \tag{3.13}$$

The initial conditions are evaluated at $(t,\tau) = (0,0)$ and are given as follows:

$$\zeta^{ij} = 0,\ i = 1,2,3,\cdots;\ j = 1,2,3,\cdots;\ \zeta^{10}_{,t} = 0 \tag{3.14a}$$

$$\zeta^{1p}_{,t} + (1-\lambda)^{-\frac{1}{2}}\zeta^{1s}_{,\tau} = 0\ ;\ \zeta^{20}_{,t} + (1-\lambda)^{-\frac{1}{2}}\mu_1'(0)\zeta^{10}_{,t} = 0 \tag{3.14b}$$

$$\zeta^{2p}_{,t} + (1-\lambda)^{-\frac{1}{2}}\zeta^{1p}_{,\tau} + (1-\lambda)^{-\frac{1}{2}}\zeta^{2s}_{,\tau} = 0,\ ;\ s = p-1,\ p = 1,2,3,\cdots \tag{3.14c}$$

We now solve (3.8), using (3.14a) and get

$$\zeta^{10}(t,\tau) = \alpha_{10}(\tau)\cos Qt + \beta_{10}(\tau)\sin Qt + B\ ;\ \alpha_{10}(0) = -B_0,\ \beta_{10}(0) = 0;\ B_0 = \frac{\lambda}{1-\lambda} \tag{3.15}$$

We substitute (3.15) into (3.9), and to ensure a uniformly valid solution in the time scale t, equate to zero the coefficients of $\cos Qt$ and $\sin Qt$ and get the following respective equations

$$\beta_{10}' - \frac{\lambda f\beta_{10}}{4(1-\lambda f)} = 0\ :\ \alpha_{10}' - \frac{\lambda f\alpha_{10}}{4(1-\lambda f)} = 0 \tag{3.16a}$$

On solving (3.16a) using the last part of (3.15), we get

$$\beta_{10}(\tau) = 0;\alpha_{10}(\tau) = -B_0\left(\frac{1-\lambda}{1-\lambda f}\right),\ \alpha_{10}'(0) = -\frac{B_0^2 f'(0)}{4};\ B'(0) = \frac{B_0 f'(0)}{1-\lambda} \tag{3.16b}$$

We now solve the remaining equation in the substitution into (3.9) and get

$$\zeta^{11}(t,\tau) = \alpha_{11}(\tau)\cos Qt + \beta_{11}(\tau)\sin Qt;\alpha_{11}(0) = 0,\beta_{11}(0) = -\frac{B_0 f'(0)(4-\lambda)}{4(1-\lambda)} \tag{3.17}$$

We next substitute relevant terms into (3.10), and to ensure a uniformly valid solution in t, equate to zero the coefficients of $\cos Qt$ and $\sin Qt$ and get the following respective equations:

$$\beta_{11}' - \frac{\lambda f\beta_{11}}{4(1-\lambda f)} = -\frac{\alpha_{10}''}{2Q(1-\lambda f)^{\frac{1}{2}}}\ :\ \alpha_{11}' - \frac{\lambda f\alpha_{11}}{4(1-\lambda f)} = 0 \tag{3.18a}$$

On solving (3.18a), we get

$$\beta_{11}(\tau)=(1-\lambda f)^{-\frac{1}{4}}\left\{(1-\lambda)^{\frac{1}{4}}\beta_{11}(0)-\int_0^\tau\frac{\alpha''_{10}(s)\,ds}{2Q(1-\lambda f(s))^{\frac{1}{4}}}\right\},\ \alpha_{11}(\tau)=0 \quad (3.18b)$$

So far we have

$$\zeta^{10}=\alpha_{10}\cos Qt+B\ ;\zeta^{11}=\beta_{11}\sin Qt \quad (3.19a)$$

We now substitute for ζ^{10} from (3.19a), into (3.11), and to ensure a uniformly valid solution in t, equate to zero the coefficient of $\cos Qt$ and get

$$\mu_1'(\tau)=-\frac{\alpha B\alpha_{10}}{(1-\lambda f)^{\frac{1}{2}}},\ \mu_1'(0)=\frac{\alpha B_0^2}{(1-\lambda)^{\frac{1}{2}}},\ \mu_1''(0)=\frac{\alpha f'(0)B_0^2(4+\lambda)}{4(1-\lambda)^{\frac{3}{2}}} \quad (3.19b)$$

The remaining equation in the substitution into (3.11) is

$$L_1\zeta^{20}=\frac{\alpha Q^2}{(1-\lambda f)}\left\{r_0+\frac{\alpha_{10}^2\cos 2Qt}{2}\right\},\ \zeta^{20}(0,0)=0,\ \zeta_{,t}^{20}(0,0)+(1-\lambda)^{-\frac{1}{2}}\mu_1'(0)\zeta_{,t}^{10}(0,0)=0 \quad (3.20a)$$

$$r_0=\frac{\alpha_{10}^2}{2}+B^2\quad ,r_0(0)=\frac{3B_0^2}{2} \quad (3.20b)$$

The solution of (3.20a,b) is

$$\zeta^{20}(t,\tau)=\alpha_{20}(\tau)\cos Qt+\beta_{20}(\tau)\sin Qt+\frac{\alpha}{(1-\lambda)}\left\{r_0-\frac{\alpha_{10}^2\cos 2Qt}{6}\right\},\ \alpha_{20}(0)=-\frac{4\alpha B_0^2}{3},\ \beta_{20}(0)=0 \quad (3.21)$$

We next substitute into (3.12) and to ensure a uniformly valid solution in t, equate to zero the coefficients of $\cos Qt$ and $\sin Qt$ and get the following respective equations

$$\beta_{20}'-\frac{\lambda f\beta_{20}}{4(1-\lambda f)}=0\ ;\ \alpha_{20}'-\frac{\lambda f\alpha_{20}}{4(1-\lambda f)}=-H(\tau) \quad (3.22a)$$

$$H(\tau)=\left[\left(\frac{1}{1-\lambda f}\right)^{\frac{1}{2}}\left\{\alpha Q\beta_{11}B+\frac{\mu_1''\alpha_{10}}{2}+\mu_1'\alpha_{10}'\right\}+Q\beta_{11}\mu_1'\right],\ H(0)=-\frac{\alpha B_0^2 f'(0)r_2}{8(1-\lambda)} \quad (3.22b)$$

$$r_2=2B_0+\frac{(4+\lambda)}{(1-\lambda)^{\frac{1}{2}}}\left\{2(1+B_0)+\frac{B_0}{(1-\lambda)^{\frac{1}{2}}}\right\} \quad (3.22c)$$

If we solve (3.22a-c), we get

$$\beta_{20}(\tau)=0\ ;\alpha_{20}(\tau)=(1-\lambda f)^{-\frac{1}{4}}\left\{(1-\lambda)^{\frac{1}{4}}\alpha_{20}(0)-\int_0^\tau(1-\lambda f(s))^{\frac{1}{4}}H(s)\,ds\right\} \quad (3.23a)$$

$$\alpha_{20}'(0)=\frac{B_0^2 f'(0)}{24(1-\lambda)^2}\{32+3r_2(1-\lambda)\} \quad (3.23b)$$

The remaining equation in the substitution into (3.12) is

$$L_1\zeta^{21}=S\sin 2Qt,\ \zeta^{21}(0,0)=0\ ,\zeta_{,t}^{21}(0,0)+(1-\lambda)^{-\frac{1}{2}}\mu_1'(0)\zeta_{,t}^{11}(0,0)+(1-\lambda)^{-\frac{1}{2}}\zeta_{,\tau}^{20}(0,0)=0 \quad (3.24a)$$

$$S=\left[\frac{2\alpha Q^2\alpha_{10}\beta_{11}}{(1-\lambda f)}\right]-\frac{2Q\alpha\left(\alpha_{10}^2\right)'}{3(1-\lambda f)^{\frac{1}{2}}}-\frac{\lambda fQ\alpha_{10}^2}{6(1-\lambda f)^{\frac{3}{2}}} \quad (3.24b)$$

On solving (3.24a,b) we get

$$\zeta^{21}(t,\tau)=\alpha_{21}(\tau)\cos Qt+\beta_{21}(\tau)\sin Qt-\frac{S\sin 2Qt}{3Q^2}\ ;\ \alpha_{21}(0)=0,\ \beta_{21}(0)\neq 0 \quad (3.25)$$

We may not particularly need the value of $\beta_{21}(0)$ in subsequent analysis. Thus far we have

$$\xi_1=\bar{\xi}\left(\zeta^{10}+\delta\zeta^{11}+\cdots\right)+\bar{\xi}^2\left(\zeta^{20}+\delta\zeta^{21}+\cdots\right)+\cdots \quad (3.26)$$

4. Maximum displacement and dynamic buckling load

The condition for the attainment of maximum displacement is

$$\xi_{1,t}(t_a,\tau_a)+\left(1-\lambda(f\tau_a)\right)^{-\frac{1}{2}}\left\{\mu_1'(\tau_a)\bar{\xi}\xi_{1,t}(t_a,\tau_a)+\delta\,\xi_{1,\tau}(t_a,\tau_a^*)\right\}=0 \qquad (4.1)$$

where t_a and τ_a are the values of t and τ respectively at maximum displacement. We let $\tilde{t}_a$ and $\hat{t}_a$ be the values of $\tilde{t}$ and $\hat{t}$ respectively at maximum displacement ξ_{1a} of $\xi_1(t,\tau)$, and now adopt the following series

$$t_a=t_0+\delta\,t_{01}+\bar{\xi}\left(t_{10}+\delta\,t_{11}+\cdots\right)+\bar{\xi}^2\left(t_{20}+\delta\,t_{21}+\cdots\right)+\cdots \qquad (4.2a)$$

$$\tilde{t}_a=\tilde{t}_0+\delta\,\tilde{t}_{01}+\bar{\xi}\left(\tilde{t}_{10}+\delta\,\tilde{t}_{11}+\cdots\right)+\bar{\xi}^2\left(\tilde{t}_{20}+\delta\,\tilde{t}_{21}+\cdots\right)+\cdots \qquad (4.2b)$$

$$\hat{t}_a=\hat{t}_0+\delta\,\hat{t}_{01}+\bar{\xi}\left(\hat{t}_{10}+\delta\,\hat{t}_{11}+\cdots\right)+\bar{\xi}^2\left(\hat{t}_{20}+\delta\,\hat{t}_{21}+\cdots\right)+\cdots \qquad (4.2c)$$

$$\tau_a=\delta\,\hat{t}_a=\delta\left\{\hat{t}_0+\delta\,\hat{t}_{01}+\bar{\xi}\left(\hat{t}_{10}+\delta\,\hat{t}_{11}+\cdots\right)+\bar{\xi}^2\left(\hat{t}_{20}+\delta\,\hat{t}_{21}+\cdots\right)\right\} \qquad (4.2d)$$

If we substitute (4.2a-d) into (4.1) and equate the coefficients of $\bar{\xi}$, $\bar{\xi}\delta$, $\bar{\xi}^2$ and $\bar{\xi}^3$, we have the following respective equations evaluated at $(t_a,\tau_a)=(t_0,0)$:

$$\zeta^{10}_{,t}=0,\quad t_{01}\zeta^{10}_{,tt}+\hat{t}_0\zeta^{10}_{,t\tau}+(1-\lambda)^{\frac{1}{2}}\zeta^{10}_{,\tau}=0,\quad t_{10}\zeta^{10}_{,tt}+\zeta^{20}_{,t}+(1-\lambda)^{\frac{1}{2}}\mu_1'(0)\zeta^{10}_{,t}=0 \qquad (4.3a)$$

$$t_{20}\zeta^{10}_{,tt}+t_{10}\zeta^{20}_{,tt}+(1-\lambda)^{\frac{1}{2}}\mu_1'(0)t_{10}\zeta^{10}_{,tt}=0 \qquad (4.3b)$$

From the first equation in (4.3a), we have

$$t_0=\frac{\pi}{Q} \qquad (4.4a)$$

Where we have taken the least nontrivial value of t_0. From the second equation in (4.3a), we have

$$t_{01}=-\frac{1}{\zeta^{10}_{,tt}}\left(\hat{t}_0\zeta^{10}_{,t\tau}+\zeta^{11}_{,t\tau}+(1-\lambda)^{\frac{1}{2}}\zeta^{10}_{,\tau}\right)\Bigg|_{(t_0,0)}$$

$$=\frac{f'(0)}{4Q^2}\left\{(4-\lambda)-(4+\lambda)(1-\lambda)^{\frac{1}{2}}\right\} \qquad (4.4b)$$

From the third equation in (4.3a) and equation (4.3b) we have

$$t_{10}=t_{20}=0 \qquad (4.4c)$$

Meanwhile if we substitute (4.2a-d) into (3.26), evaluated at $(t,\tau)=(t_a,\tau_a)$, we have the following, evaluated at $(t_a,\tau_a)=(t_0,0)$

$$\xi_{1a}=\bar{\xi}\left\{\zeta^{10}+\delta\left(\zeta^{11}+\hat{t}_0\zeta^{10}_{,\tau}\right)\right\}+\bar{\xi}\left[\zeta^{20}+\delta\left\{\hat{t}_{10}\zeta^{10}_{,\tau}+t_{01}\zeta^{20}+\hat{t}_0\zeta^{20}_{,\tau}+\zeta^{21}\right\}\right]$$
$$+O\left(\bar{\xi}\delta^2\right)+O\left(\bar{\xi}^2\delta^2\right) \qquad (4.5)$$

Terms not included in (4.5) will automatically vanish on substitution.

We thus need to evaluate the terms t_0, t_{01}, t_{10} and t_{20}, some of which will be used later. From the second term in (3.4), we have

$$\tilde{t}_a=\int_0^{\hat{t}_a}\left(1-\lambda f(\delta s)\right)^{\frac{1}{2}}ds=(1-\lambda)^{\frac{1}{2}}\int_0^{\hat{t}_a}\left[1-\frac{\lambda}{2(1-\lambda)}\left\{(\delta s)f'(0)+\frac{(\delta s)f''(0)}{2}\right\}\right.$$
$$\left.-\frac{1}{8}\left(\frac{\lambda}{1-\lambda}\right)^2\left\{(\delta s)f'(0)+\frac{(\delta s)f''(0)}{2}\right\}^2+\cdots\right]$$
$$=(1-\lambda)^{\frac{1}{2}}\left[\hat{t}_a-\frac{\lambda}{2(1-\lambda)}\left\{\frac{\delta\,\hat{t}_a^2}{2}f'(0)+\cdots\right\}\right]+O\left(\hat{t}_a^3\right) \qquad (4.6)$$

If we substitute into (4.6) for $\tilde{t}_a$ and $\hat{t}_a$ from (4.2b,c) and equate the coefficients of $O(1)$, δ, $\bar{\xi}$ and $\bar{\xi}^2$, we get the following respective values

$$\tilde{t}_0 = (1-\lambda)^{\frac{1}{2}} \hat{t}_0 \ , \ \tilde{t}_{01} = (1-\lambda)^{\frac{1}{2}} \left\{ \hat{t}_{01} - \frac{\lambda \hat{t}_0}{4(1-\lambda)} \right\} , \tilde{t}_{10} = (1-\lambda)^{\frac{1}{2}} \hat{t}_{10} \ , \tilde{t}_{20} = (1-\lambda)^{\frac{1}{2}} \hat{t}_{20} \quad (4.7)$$

If we evaluate the last equation in (3.4a) at maximum values, we have

$$t_a = \tilde{t}_a + \bar{\xi}\{\mu_1'(0)\hat{t}_a + \cdots\} \qquad (4.8)$$

On substituting for t_a, $\tilde{t}_a$ and $\hat{t}_a$ into (4.8) from (4.2a-c) and equating the coefficients of δ, $\bar{\xi}$, $\delta\bar{\xi}$ and $\bar{\xi}^2$, we get the following respective values

$$\hat{t}_0 = \frac{\pi}{Q(1-\lambda)^{\frac{1}{2}}} \ , \ \hat{t}_{01} = \frac{t_{01}}{(1-\lambda)^{\frac{1}{2}}} + \frac{\lambda \hat{t}_0}{4(1-\lambda)} \ , \ \hat{t}_{10} = -\frac{\alpha B_0^2 \hat{t}_0}{(1-\lambda)} \ , \ \hat{t}_{20} = -\frac{\alpha B_0^2 \hat{t}_{10}}{(1-\lambda)} \qquad (4.9)$$

If we simplify (4.5) using all the relevant terms so far evaluated, we have

$$\xi_{1a} = 2B_0\bar{\xi}\{1+\delta f' A_{11}(\lambda)\} + \frac{8\alpha B_0^2}{3(1-\lambda)}\{1+\delta f' A_{22}(\lambda)\} \qquad (4.10a)$$

$$A_{11}(\lambda) = \frac{\hat{t}_0(4+\lambda)}{24(1-\lambda)} \ , \ A_{22}(\lambda) = \frac{3(4+\lambda)\hat{t}_{10}}{32B_0\alpha} + t_{01} + \frac{\hat{t}_0 r_3}{64(1-\lambda)} \ , \ r_3 = 4(4-\lambda) - 3r_2(1-\lambda) \quad (4.10b)$$

Meanwhile if the values of t, τ and $\hat{t}$ for ξ_0 to attain the maximum ξ_{0a} are respectively given as t_c, τ_c and $\hat{t}_c$, then, from (3.1), using (3.6), we have

$$\xi_{0a} = \lambda f(\delta \hat{t}_c) + \frac{K_0\bar{\xi}^2}{\lambda_c}\left[(\varsigma^{10})^2 + 2\varsigma^{10} + 2\delta\varsigma^{11}\{\varsigma^{10} + 1\}\right] + \cdots \qquad (4.11)$$

The condition for ξ_0 to attain ξ_{0a} is

$$\xi_{0,t}(t_c,\tau_c) + (1-\lambda(f\tau_c))^{\frac{1}{2}}\{\mu_1'(\tau_c)\bar{\xi}\xi_{1,t}(t_c,\tau_c) + \delta\xi_{0,\tau}(t_c,\tau_c)\} = 0 \qquad (4.12)$$

We let $\tilde{t}_c$ be the value of $\tilde{t}$ at the maximum ξ_{0a} and now assume the following asymptotic series

$$t_c = T_0 + \delta T_{01} + \bar{\xi}(T_{10} + \delta T_{11} + \cdots) + \bar{\xi}^2(T_{20} + \delta T_{21} + \cdots) + \cdots \qquad (4.13a)$$

$$\tilde{t}_a = \tilde{T}_0 + \delta \tilde{T}_{01} + \bar{\xi}(\tilde{T}_{10} + \delta \tilde{T}_{11} + \cdots) + \bar{\xi}^2(\tilde{T}_{20} + \delta \tilde{T}_{21} + \cdots) + \cdots \qquad (4.13b)$$

$$\hat{t}_c = \hat{T}_0 + \delta \hat{T} + \bar{\xi}(\hat{T}_{10} + \delta \hat{T}_{11} + \cdots) + \bar{\xi}^2(\hat{T}_{20} + \delta \hat{T}_{21} + \cdots) + \cdots \qquad (4.13c)$$

$$\tau_a = \delta \hat{t}_c = \delta\{\hat{T}_0 + \delta \hat{T} + \bar{\xi}(\hat{T}_{10} + \delta \hat{T}_{11} + \cdots) + \bar{\xi}^2(\hat{T}_{20} + \delta \hat{T}_{21} + \cdots) + \cdots\} \qquad (4.13d)$$

On substituting (4.13a-d) into (4.11), we have

$$\xi_{0a} = \lambda(1 + \delta \hat{T}_0 f'(0)) + \lambda\bar{\xi}\delta f'(0)\hat{T}_{10} + \bar{\xi}^2 \left[(\varsigma^{10})^2 + 2\varsigma^{10} + 2\delta\{(\varsigma^{10} + \varsigma^{10} + \varsigma\}\hat{T}_0 \right.$$

$$\left. + \frac{f'(0)\hat{T}_{20}}{2} + \varsigma^{11}\left(\varsigma^{10} + \frac{K_0}{\lambda_c} \right) \right]\Bigg|_{(1,0)} + \cdots \qquad (4.14)$$

where terms not included in (4.14) will vanish on substitution. Thus, we need to determine $\hat{T}_0$, $\hat{T}_{10}$ and $\hat{T}_{20}$. Analysis using (4.12) shows that ξ_1 and ξ_0 attain their maxima at same values of the independent variables. Hence we have $T_0 = t_0$, $\hat{T}_0 = \hat{t}_0$, $T_{10} = t_{10}$, $\hat{T}_{10} = \hat{t}_{10}$ and $\hat{T}_{20} = \hat{t}_{20}$. Thus on simplifying (4.14) we have

$$\xi_{0a} = \lambda(1 + \delta \hat{T}_0 f'(0)) + \lambda\bar{\xi}\delta f'(0)\hat{T}_{10} + \frac{K_0\bar{\xi}^2}{\lambda_c}\left[4B_0(B_0 + 1) + 2\delta f'(0)\left\{ \frac{B_0(4+\lambda)(2B_0+1)\hat{t}_0}{4(1-\lambda)} \right. \right.$$

$$\left. \left. + \frac{\lambda\lambda_c \hat{T}_{20}}{2K_0} \right\} \right] + \cdots \qquad (4.16)$$

Using the second equation in (2.8), the net maximum displacement ξ_a now becomes (from (4.10a,b) and (4.16))

$$\xi_m = \xi_a - \lambda\left(1+\delta\hat{T}_0 f'(0)\right) = C_1\bar{\xi} + C_2\bar{\xi}^2 + \cdots \qquad (4.17a)$$

$$C_1 = 2B_0\left(1+\delta f'(0)A_{33}(\lambda)\right), \mathrm{C}_2 = \frac{8\alpha \mathrm{B}_0^2}{3(1-\lambda)}\left[1 + A_{44}(\lambda) + \delta f'(0)A_{55}(\lambda)\right] \qquad (4.17b)$$

$$A_{33}(\lambda) = \left(A_{11}(\lambda) + \frac{\lambda\hat{T}_{10}}{2B_0}\right), A_{44}(\lambda) = \frac{3(1-\lambda)K_0}{2B_0\lambda_c}(B_0+1) \qquad (4.17c)$$

$$A_{55}(\lambda) = A_{22}(\lambda) + \frac{3(1-\lambda)K_0}{4\alpha B_0^2\lambda_c}\left[A_{22}(\lambda) + \frac{4\alpha B_0^2\lambda_c}{3(1-\lambda)K_0}\left\{\frac{B_0(4+\lambda)(2B_0+1)\hat{T}_0}{4(1-\lambda)} + \frac{\lambda\lambda\hat{T}_{20}}{2K_0}\right\}\right] \qquad (4.17d)$$

We note that the term $\lambda\left(1+\delta\hat{T}_0 f'(0)\right)$ is $O(1)$ in $\bar{\xi}$. To determine the dynamic buckling λ_D, we use an equivalent form of the first of (2.8) which now becomes $\frac{d\lambda}{d\xi_m} = 0$. As in Amazigo and Ette (1987), we note that before invoking the above maximization, we first have to reverse the series (4.17a) in the form

$$\bar{\xi} = d_1\xi_m + d_2\xi_m^2 + \ldots \qquad (4.19a)$$

By substituting in (4.19a) for ξ_m, and equating the coefficients of $\bar{\xi}$ and ξ_m^2, we get

$$d_1 = \frac{1}{C_1}, \mathrm{d}_2 = -\frac{C_2}{C_1^3} \qquad (4.19b)$$

The maximization $\frac{d\lambda}{d\xi_m} = 0$ easily follows ,through (4.19a), to give

$$\xi_{mD} = \xi_m(\lambda_D) = \frac{d_1}{2d_2} = \frac{C_1^2}{2C_2} \qquad (4.19c)$$

where ξ_{mD} is the value of ξ_m at buckling (i.e. at $\lambda = \lambda_D$). On evaluating (4.19a) at $\lambda = \lambda_D$, we get

$$\bar{\xi} = \frac{C_1(\lambda_D)}{4C_2(\lambda_D)} \qquad (4.19d)$$

If we substitute into (4.19d) for $C_1(\lambda_D)$ and $C_2(\lambda_D)$ from (4.17), we get

$$(1-\lambda_D)^2 = \frac{16\alpha\bar{\xi}\lambda_D}{3}\left[\frac{1 + A_{44}(\lambda_D) + \delta f'(0)A_{55}(\lambda_D)}{1+\delta f'(0)A_{33}(\lambda_D)}\right] \qquad (4.20)$$

5. Analysis of result and conclusion

If we neglect the inertia terms in (2.1) and (2 2) or in (2 4) and (2.5) and set $f(T) \equiv 1$, we have the equivalent equations characterizing the static theory. The static buckling load λ_S is obtained from the maximization $\frac{d\lambda}{d\xi_1} = 0$ and this yields

$$(1-\lambda_S)^2 = 4\alpha\lambda_S\bar{\xi}\left(1 + \frac{K_0\bar{\xi}}{\alpha\lambda_c}\right) \qquad (5.1)$$

If we eliminate the imperfection parameter $\bar{\xi}$ from (5.1), using (4.20), we easily get

$$\left(\frac{1-\lambda_D}{1-\lambda_S}\right)^2 = \frac{4}{3}\left(\frac{\lambda_D}{\lambda_S}\right)\left[\frac{1 + A_{44}(\lambda_D) + \delta f'(0)A_{55}(\lambda_D)}{1 + \delta f'(0)A_{33}(\lambda_D)}\right] \times$$

$$\left[1 + \frac{3K_0}{16\alpha^2\lambda_c\lambda_D}\left\{\frac{1+\delta f'(0)A_{33}(\lambda_D)}{1 + A_{44}(\lambda_D) + \delta f'(0)A_{55}(\lambda_D)}\right\}\right]^{-1} \qquad (5.2)$$

If the loading were a step load, then $f^{(n)}(0) = 0$, for $n \geq 1$ so that the step loading results corresponding (4.20) and (5.1) are given respectively as

$$(1-\lambda_D)^2 = \frac{16\alpha\bar{\xi}\lambda_D}{3}(1+A_{44}(\lambda_D)) \qquad (5.3a)$$

$$\left(\frac{1-\lambda_D}{1-\lambda_S}\right)^2 = \frac{4}{3}\left(\frac{\lambda_D}{\lambda_S}\right)(1+A_{44}(\lambda_D))\left[1+\frac{3K_0}{16\alpha^2\lambda_C\lambda_D\{1+A_{44}(\lambda_D)\}}\right]^{-1} \qquad (5.3a)$$

Thus , given the static buckling load λ_S, we can easily calculate the dynamic buckling load λ_D(and vice versa) ,without necessarily performing the arduous task of repeating the entire process for different imperfection parameters. If, in the step loading case, we neglect the second spring with spring constant K_0 (Budiansky's problem, (1966)), we get the following results ,from (4.20) and (5.1) as

$$(1-\lambda_D)^2 = \frac{16\alpha\bar{\xi}\lambda_D}{3} \text{ and } (1-\lambda_S)^2 = 4\alpha\lambda_S\bar{\xi} \qquad (5.3c)$$

On eliminating the imperfection parameter in (5.3a) we have

$$\frac{3}{4}\left(\frac{1-\lambda_D}{1-\lambda_S}\right)^2 = \left(\frac{\lambda_D}{\lambda_S}\right) \qquad (5.3d)$$

Equation (4.20) is valid provided $|A_{44}(\lambda_D)+\delta f'(0)A_{55}(\lambda_D)|<1$, and $|\delta f'(0)A_{33}(\lambda_D)|<1$. In addition to the above two inequalities, equation (5.2) is also valid if $\left|\frac{3K_0}{16\alpha^2\lambda_C\lambda_D}\left\{\frac{1+\delta f'(0)A_{33}(\lambda_D)}{1+A_{44}(\lambda_D)+\delta f'(0)A_{55}(\lambda_D)}\right\}\right|<1$.

Danielson (1969) obtained the following results for the step loading case:

$$\left(\frac{\lambda_D}{\lambda_S}\right) = \frac{3}{4}\left(\frac{1-\lambda_D}{1-\lambda_S}\right)^2, \text{ for } 0<\left(\frac{\omega_1}{\omega_0}\right)<\frac{1}{2} \qquad (5.4a)$$

$$\left(\frac{\lambda_D}{\lambda_S}\right) = \frac{\frac{1}{6}\left\{4-\left(\frac{\omega_0}{\omega_1}\right)^2\right\}}{\lambda_S+\frac{10}{9}\left(\frac{\omega_1}{\omega_0}\right)(1-\lambda_D)^2}, \text{ for } \frac{1}{2}<\left(\frac{\omega_1}{\omega_0}\right)<1 \qquad (5.4b)$$

We remark that Danielson used the method of Mathieu-type of instability, which ,as noted by Budiansky (1966, page 100), is always associated with many cycles of oscillation, as opposed to just one shot of oscillation that normally triggers off dynamic buckling Thus, we expect our results (5.3a,b) , for the step loading case , and other results such as (4.20) and (5.2) , to be much more representative of the buckling process than (5.4a,b) .We must however note the exact correspondence between our result (5.3d) and Danielson's result (5.4a) in the interval $0<\left(\frac{\omega_1}{\omega_0}\right)<\frac{1}{2}$.However the method of Mathieu-type of instability used by Danielson cannot account for explicitly time-dependent loading history in which our method has an overwhelming advantage .

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