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ESTIMATION OF THE OUTLIER-FREE AND OUTLIER-CONTAMINATED TRANSFER FUNCTION MODELS

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ABSTRACT

This paper explored the Transfer Function (TF) Models for both the outlier-free (OF) and outlier contaminated (OC) series. A TF model of order q,r [TF (q,r)] was defined for the two observable stationary time series and the contribution of outlier input series in TF model as well as its effects on output series generated was examined. It was observed that outliers affect significantly the estimates of the models Apart from this, the model residual variance was affected too. The test criterion was also seriously jeopardized as a tool for measuring the adequacy of fit because outlier series are embedded in its computation.

KEYWORDS: Transfer function; outlier; outlier-free; outlier-contaminated; model residual variance.

1. INTRODUCTION

Time series data often contain outliers or observations that are distinct from most of the other observations which have an effect on parameter estimates and more importantly lead to inaccurate forecasts (Chang (1982) and Tsay (1984)). A common approach to deal with outliers in a time series is to identify the locations and the types of outliers and then use intervention models discussed by Box and Tiao (1975) to accommodate the outlier effects. This approach requires iteration between stages of outlier detection and estimation of an intervention model. Wetherill (1986) in data exploration pointed out that two broad questions may arise and these must be carefully distinguished. One problem is how to react to the outliers and what principles and methods can be used to support rejecting them, adjusting their values or leaving them unaltered prior to processing the principal mass of data. The next problem is what to do with these outliers once they are located.

Outliers can take several forms in time series. Fox (1972) proposed the formal definitions and a classification of outliers in time series context. He proposed a classification of time series outliers to Type I and Type II based on an autoregressive model. These two types have later been renamed as additive and innovational outliers (Muirhead (1986)).

For a properly deduced stationary process, let X_t be the observed series and Z_t be the outlier free series. Consider a familiar time series model (Tsay (1986)).

$$\Pi(B) Z_t = a_t$$

where

$$\Pi(B) = 1 - \Pi_1 B - \Pi_2 B^2 - \dots$$

$\{a_t\}$ is a sequence of identically, independently distributed normal variables with zero mean and variance σ^2. Assume that $\{Z_t\}$ follows a general ARMA (p, q) model, $\phi(B)Z_t = \theta(B)\, a_t$

where

$$\phi(B) = 1 - \phi_1 B - \dots - \phi_p B^p$$
$$\theta(B) = 1 - \theta_1 B - \dots - \theta_q B^q$$

and

$\{a_t\}$ is a sequence of white noise, independent and identically distributed (Tsay (1986)).

The models commonly employed on the outlier-free time series Z_t are the Additive Outlier (AO) and the Innovational Outlier (IO) which are defined respectively of a single outlier for a simple case as

$$X_t = \begin{cases} Z_t & t \neq T \\ Z_t + D & t = T \end{cases}$$
$$= Z_t + D\xi_t^{(T)}$$

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$$= \frac{\theta(B)}{\phi(B)} a_t + D\xi_t^{(T)} \quad (1)$$

where

$$\xi_t^{(T)} = \begin{cases} 1 & , \ t = T \\ 0 & , \ t \neq T \end{cases}$$

is the indicator variable representing the presence or absence of an outlier at time T (Fox (1972), Abraham and Box (1979), Denby and Martin (1979) and Chang and Tiao (1983)).
An innovational outlier (IO) model is defined as clearly pointed out by Fox (1972) and Chang and Tiao (1983)

$$X_t = Z_t + \frac{\theta(B)}{\phi(B)} D\xi_t^{(T)} \quad (2)$$

The AO affects the level of the Tth observation whereas an IO affects all observations $X_T, X_{T+1}, \ldots$ beyond time T through the memory of the system described by $\theta(B)/\phi(B)$. The AO can be regarded as a gross error model (Tsay (1986)).
In general, Tsay (1986) showed that the presence of more than one outlier of various types in a model is specified by

$$X_t = \sum_{k=1}^{n} V_k(B) D_k \xi_t^{(T)} + Z_t \quad (3)$$

where

$Z_t = \theta(B)\phi^{-1}(B)\, a_t$
$V_k(B) = 1$ for AO model
$V_k(B) = \dfrac{\theta(B)}{\phi(B)}$ for IO model

at time $t = T_k$ and n is the number of outliers.
Fox (1972) made no methodological proposals to distinguish between the two basic types of outliers. This is remedied in the work of Muirhead (1986) who presents a test of discordancy for a single outlier of unknown type and proceeds to examine the properties of an appealing likelihood-ratio-based rule for distinguishing whether the outlier is of AO or IO type. He also compares this rule with corresponding Bayesian procedure.

2. OUTLIER BASED TRANSFER FUNCTION MODEL

Box and Jenkins (1976) expressed mathematically a basic transfer relationship for a bivariate stochastic process for X_t and Y_t as:

$$\phi(B)Y_t = \theta(B)X_t + \varepsilon_t \quad (4)$$

or

$$Y_t = \frac{\theta(B)}{\phi(B)} X_t + \varepsilon_t^*$$

where

$$\theta(B) = 1 - \sum_{j=1}^{q} \theta_j B^j$$

$$\phi(B) = 1 - \sum_{j}^{r} \phi_j B^j$$

and

$$\varepsilon_t^* = \phi^{-1}(B)\, \varepsilon_t$$

The two polynomial functions in B, $\phi^{-1}(B)\ \theta(B)$ is the model's transfer function, ε_t^* is the inherent uncontrollable random effects which contribute to the transformation of X_t to Y_t.

Equation (4) is said to be a transfer function model denoted by TF (q, r, d) in which $\phi(B)$ is of order r, $\theta(B)$ is of order q and the series in the model requires differencing d times to attain stationarity (Box and Jenkins (1976)).

In contrast to the series X_t, X_{t-b} indicates the extent of delay (measured by b) before a response from Y_t manifests. The resultant impulse response function denoted by V(B) is analogous to the transfer function, since from the expression

$$Y_t = V(B)X_{t-b} + \varepsilon_t^* \quad (5)$$

where

$$V(B) = V_o + V_1B + \ldots + V_qB^q$$

When $V(B) = \phi^{-1}(B)\ \theta(B)$, the impulse response function model in equation (5) becomes a transfer function model (Box and Jenkins (1976)).

There exists similarities between a linear regression and transfer function model. As a result of this, the detection of outliers and other diagnostics based on residuals have been widely used in linear regression. Box and Jenkins (1976) and Chatfield (1980) suggested that the stochastic properties of residuals must be examined for the adequacy of fit of the models. To check whether the errors are normally distributed, one can construct a histogram of the standardized residuals $a_t/\hat{\sigma}_a$, and compare it with the standard normal distribution using the chi-square goodness of fit test or even the Tukey's simple five-number summary (Tukey (1960)). To check whether the residuals are white noise we compute the sample autocorrelation function of the residuals to see whether they do not form any pattern and are all statistically insignificant (Box and Jenkins (1976) and Chatfield (1980)).

Let us assume that the input series X_t follow an ARMA process of order (q, r) and suppose the inverse autocorrelation function in ε_t is characterized by

$$\phi(B)X_t = \theta(B)\ \varepsilon_t \quad (6)$$

where

$$\phi(B) = \sum_{j=1}^{q} \phi_j B^j$$

$$\theta(B) = \sum_{j=1}^{r} \phi_j B^j$$

$\phi_o = 1$, $\theta_o = 1$ and ε_t is a white noise process

$\varepsilon_t = \dfrac{\phi(B)}{\theta(B)} X_t$ is often called the prewhitened input series (Box and Jenkins (1976)).

Applying the prewhitening transformation to the output series Y_t, we obtain a filtered output series

$$\varepsilon_t = \frac{\phi(B)}{\theta(B)} X_t .$$

The combined dynamic-disturbance model is given by

$$\phi(B)\ Y_t = \theta\ (B)\ X_t + \frac{\phi(B)}{\theta(B)} X_t \quad (7)$$

Pierce (1972) has shown that

$$Y_t = \frac{\theta(B)}{\phi(B)} Z_t + \frac{\theta(B)}{\phi(B)} D\xi_t^{(T)} + \varepsilon_t^* \quad (8)$$

$$= \Pi^*(B) Z_t + K_t$$

where

$$\Pi^*(B) = \frac{\theta(B)}{\phi(B)}$$

$$\varepsilon_t\ \ \phi(B)^{-1}\varepsilon_t$$

and

$$K_t = \Pi^*(B) D\xi_t^{(T)} + \varepsilon_t^* \quad (9)$$

is the residual of the outlier induced model defined in equation (8).

Following Tsay (1986), the magnitude of the outliers based on the least squares theory is

$$D^* = \eta^2\ \alpha\ (B)\ J_t$$

where

$J_t = \alpha(B) X_t$

$\alpha(B) = \phi(B)/\theta(B)$

$= 1 - \alpha_1 B - \alpha_2 B^2 - \ldots$

(obtained after fitting ARMA model to the outlier-free series Z_t)

$\eta^2 = (1 + \alpha_1^2 + \ldots + \alpha^2_{n-k})^{-1}$

k = number of fitted lags.

The variance of the estimate of D^* is

$$\text{Var}(D^*) = \eta^2 \sigma_a^2$$

where σ_a^2 is the variance of a_t for an outlier-free series.

Hence

$$\text{Var}(K_t) = \eta^2 \sigma_a^2 + \sigma_\varepsilon^2 \quad (10)$$

where

σ_ε^2 is the variance of ε_t

The variability in the residual model for an outlier-contaminated series is increased by $\eta^2 \sigma_a^2$ as is evident from equation (10) and the precision of the output generated can be measured through the variance of K_t.

For an outlier-contaminated (OC) and outlier-free (OF) models, we assume that they follow TF (q, r) models and that the error term in both models is white noise and define the models as

$$\text{OC:} \quad Y_t = \phi_1 Y_{t-1} + \ldots + \phi_q Y_{t-q} + \omega_1 Z_{t-1} + \ldots + \omega_r Z_{t-r} + \omega_1 D_{t-1} + \ldots + \omega_r D_{t-r} + \varepsilon_t \quad (11)$$

$$\text{OF:} \quad Y_t = \phi_1 Y_{t-1} + \ldots + \phi_q Y_{t-q} + \omega_1 X_{t-1} + \ldots + \omega_r X_{t-r} + \varepsilon_t \quad (12)$$

Olewuezi and Shangodoyin (2005) used the linear regression approach to obtain sets of q + r linear simultaneous equation for the inverse auto covariance function.

3. EMPIRICAL ILLUSTRATION

To examine the contribution of outlier input series in TF model specification as well as its effects on the output series generated, an outlier input series was assumed and Tsay (1986) detection techniques were used to observe the timings of the outliers. The estimates of the coefficients were computed with their variances. A comparison is made between OF and OC transfer function models. Series 1 is the sales data with leading indicator published by Box and Jenkins (1976), Series 2 is the simulated data of size 100. The simulation method is as follows (Olewuezi (2007)).

(a) Obtain normal random deviates using the random normal generator formula from the excel package.

(b) Repeat step (a), one hundred times.

(c) Obtain the mean μ and the standard deviation σ of the series.

(d) Standardize the series using

$$Z = \frac{X - \mu}{\sigma}$$

where

X_t is the generated observation.

The simulation has two purposes to compare the behaviour of inverse autocorrelation function (IACF) for outlier-free and outlier-contaminated data and to show that when the outlier effect is moderate, the IACF can still specify the underlying model for Z_t and the employed outlier detection procedure can then identify the existing outlier. By the model suggested by Olewuezi and Shangodoyin (2005), the inverse autocorrelations for the series were obtained. Also, using the outlier detection techniques by Tsay (1986), the timings and the number of outliers were obtained as shown in Table 1. Table 1 shows the estimates of the models fitted with their standard errors.

Table 1: Estimates of the models fitted with their standard errors in bracket.

Type of series	1		2	
	without outliers	with outliers	without outliers	with outliers
Number of outliers	-	25	-	10
Timing	-	2, 5, 6, 9, 12, 15, 16, 25, 30, 32, 37, 60, 65, 100, 102, 104, 111, 113, 119, 120, 130, 132, 134, 139, 142	-	1, 5, 10, 15, 20, 37, 39, 52, 59, 70
σ Estimate σ_1 σ_2	 0, 332 (0.121) 0.253 (0.182)	 1.952 (0.230) 0.352 (0.248)	 0.821 (0.013)	 0.935 (0.128)
ω Estimate ω_0 ω_1 ω_2	 1.230 (0.032) 0.653 (0.210) 1.367 (0.238)	 -0.321(0.520) -0.951(0.321) 1.023 (0.421)	 0.310 (0.024) -0.119 (0.101)	 -0.521 (0.102) 0.237 (0.203)
Prob (F)	0.000	0.011	0.081	0.041
Variance of the residual	3.125	4.351	0.560	1.006

Series 1

The series was differenced once before attaining stationarity. A TF (2, 1, 2) models were fitted for both the OF and OC models.

For the OC we fitted

$$Y_t = 1.952Y_{t-2} + 0.352Y_{t-1} + 0.321X_t - 0.951X_{t-1} + 1.023X_{t-2} + \varepsilon_t$$

For the OF we fitted

$$Y_t = 0.332Y_{t-2} + 0.253Y_{t-1} + 1.230X_t + 0.653X_{t-1} + 1.367X_{t-2} + \varepsilon_t$$

The model residual variance with outliers is 1.39 multiple of the model residual variance without outliers. The magnitude of the standard error of the estimates of ϕ_2, ϕ_1, ω_0, ω_1 and ω_2 are smaller for the OF model than what was obtained for the contaminated model. The test statistics indicate that both models are adequate and the power of the test for the model without outlier is better than that obtained for the outlier contaminated model.

Series 2

This is the simulated series of size 100. A TF (1, 1) was fitted to the series when the input are

OC: $Y_t = 0.935Y_{t-1} - 0.521X_t + 0.237X_{t-1} + \varepsilon_t$

and

OF: $Y_t = 0.821Y_{t-1} + 0.310X_t - 0.119X_{t-1} + \varepsilon_t$

The residual variance of the contaminated series is about 1.796 multiple of that obtained for the model without outliers. The standard errors of the estimates ϕ_1, ω_0 and ω_1 for the OF model are greater than that obtained for the OC model. The test statistics indicate that both models are adequate.

4. IMPLICATIONS OF THE RESULT

Outliers affect significantly the estimates of the model. Understanding the source and reason of an outlier can provide insight into the series and hence suggest an appropriate method for handling the outlier. The model residual is also affected. Of course, this will have a combined effect on the precision of the output generated. These biases can be severe and depend on, besides the obvious attributes like the number, type, magnitude and position of the outliers, the underlying model, its autocorrelation structure and hence the distribution of the estimates of the TF models. The finding has revealed that the standard error of the model is also affected. The method cannot reveal what caused the series to behave inconsistently, it can often identify those observations that deserve special treatments. Consequently, a substantial reduction in residual mean square was obtained. In series of short to moderate lengths, often the presence of a single outlier will result in a true AR model being falsely identified as a MA or an ARMA model and the identified lag lengths (q or r) will also be wrong. The presence of outliers may then cause biases in TF modeling and this can jeopardize the functions as model identification tools. As a suggestion, a check of the series is very necessary for the presence of outliers before modeling.

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