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PERIODIC SOLUTIONS IN A NONLINEAR FOURTH ORDER DIFFERENTIAL EQUATION II

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ABSTRACT

The hypothesis $\chi(m) \neq 0$ has the following implications

$$\chi(m) = m^4 - a_2 m^2 + a_4 \neq 0$$

or

$$a_3 - a_1 m^2 \neq 0$$

However the attention of scholars have been on $\chi(m) = m^4 - a_2 m^2 + a_4 \neq 0$, which implies

$$a_4 - \tfrac{1}{4}a_2^{\,2} > 0 \text{ or } a_4 - \tfrac{1}{4}a_2^{\,2} < 0.$$

But the second condition $a_3 - a_1 m^2 \neq 0$ has been given little or no attention by scholars. In this paper, an existence result has been obtained using the alternative condition

$$\chi(m) = a_3 - a_1 m^2 \neq 0 \text{ or } m^2 \neq a_1^{-1} a_3$$

along side with other hypotheses.

KEYWORDS: Nonlinear ODE, boundary value problem, a priori bound, fixed point technique.

1. INTRODUCTION

Consider the nonlinear differential equation

$$x^{(4)} + f(\dddot{x}) + g(\ddot{x}) + h(\dot{x}) + a_4 x = P(t, x, \dot{x}, \ddot{x}, \dddot{x}) \tag{1}$$

with boundary conditions

$$D^{(r)}x(0) = D^{(r)}x(2\pi),\ r = 0,1,2,3,\ D = \frac{d}{dt} \tag{2}$$

where a_4 is a constant, $f = f(\dddot{x})$, $g = g(\ddot{x})$, $h = h(\dot{x})$, $P = P(t, x, \dot{x}, \ddot{x}, \dddot{x})$ are continuous functions with P 2π periodic in t.

In a special case, consider the constant coefficients equation

$$x^{(4)} + a_1\dddot{x} + a_2\ddot{x} + a_3\dot{x} + a_4 x = 0 \tag{3}$$

with the corresponding nonhomogeneous equation

$$x^{(4)} + a_1\dddot{x} + a_2\ddot{x} + a_3\dot{x} + a_4 x = P(t, x, \dot{x}, \ddot{x}, \dddot{x}) \tag{4}$$

both (3) and (4) subject to the boundary condition (2). The auxiliary equation

$$r^4 + a_1 r^3 + a_2 r^2 + a_3 r + a_4 = 0$$

of (3) has a root of the form $r = im$ (m an integer) if the equation

$$m^4 - a_2 m^2 + a_4 = 0 \text{ and } m(a_3 - a_1 m^2) = 0 \tag{5}$$

are satisfied simultaneously Ezeilo (1979). The boundary value problem (3) – (2) has no nontrivial solutions if either

$$\chi(m) = m^4 - a_2 m^2 + a_4 \neq 0 \tag{6}$$

or

$$a_3 - a_1 m^2 \neq 0 \text{ or } m^2 \neq a_1^{-1} a_3 \tag{7}$$

The equation (6) in its extended forms to nonlinear term has been applicable in the hypotheses for existence of periodic solutions of a fourth order ordinary differential equations. For instance, see Ezeilo

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1979), (1999), (2000), Ezeilo and Tejumola (2001), Ogbu (2006), (2007), Tiryaki (1990) and Tejumola 2006).

In this paper, our interest is on (7), which is new in the literature. Thus, we have the following

heorem 1

Suppose in addition to the basic assumptions on f, g, h, and P

i) There exist a_1, a_3 constants such that

$$\frac{f(u)}{u} \geq a_1,\ u \neq 0 \tag{8}$$

ii) The function $h(x)$ is such that

$$|h'(\dot{x})| \leq a_3 \tag{9}$$

iii) The function P is bounded and 2π periodic in t.

hen equations (1) – (2) have at least one 2π periodic solution for arbitrary $g(z)$ and a_4.

emark: This is an extension of Tejumola result for the equation $x^{(4)} + g_1\dddot{x} + g_2\ddot{x} + g_3\dot{x} + b_4 x = P(t, x, \dot{x}, \ddot{x}, \dddot{x})$ or $P_2 \neq 0$ [see Tejumola 2006].

. GENERAL COMMENTS ON SOME NOTATIONS

Throughout the proof which follows, the capitals C_1, C_2, C_3, ... represent positive constants whose nagnitude depend at most on a_4, f, g, h *and* P. The constants C_1, C_2, C_3, ... retain their identities hroughout the proof of theorem 1. The symbols $|.|_\infty$, $|.|_1$, and $|.|_2$ in respect of the mappings $[0: 2\pi] \to \Box$ hall have their usual meaning $|\dot{\theta}|_\infty = \max_{0 \leq t \leq 2\pi} |\theta(t)|$, $|\theta|_1 = \int_0^{2\pi} |\theta(t)| dt$, $|\theta|_2 = \left(\int_0^{2\pi} \theta^2(t) dt \right)^{1/2}$

. PROOF OF THEOREM 1

The proof of theorem 1 is by the Leray-Schauder fixed point technique (Leray and Schauder, 1934) nd we shall consider the parameter λ dependent equation, *($0 \leq \lambda \leq 1$)*

$$x^{(4)} + f_\lambda(\dddot{x}) + \lambda g(\ddot{x}) + h_\lambda(\dot{x}) + a_4 x = \lambda P \tag{10}$$

vhere

$$f_\lambda(\dddot{x}) = (1 - \lambda) a_1 \dddot{x} + \lambda f(\dddot{x})$$

$$h_\lambda(\dot{x}) = (1 - \lambda) a_3 \dot{x} + h(\dot{x})$$

y setting

$$\dot{x} = y,\ \dot{y} = z,\ \dot{z} = u,\ \dot{u} = -f_\lambda(u) - \lambda g(z) - h_\lambda(y) - a_4 x + \lambda P \tag{11}$$

ne equation (10) can be written compactly in matrix form

$$\dot{X} = AX + \lambda F(X, t) \tag{12}$$

vhere

$$X = \begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix}, A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_4 & -a_3 & 0 & -a_1 \end{pmatrix}, F = \begin{pmatrix} 0 \\ 0 \\ 0 \\ Q \end{pmatrix} \tag{13}$$

vith $Q = P - f(u) + a_1 u - g(z) - h(y) + a_3 y$

Note that equation (10) reduces to a linear equation

$$x^{(4)} + a_1 \dddot{x} + a_3 \dot{x} + a_4 x = 0 \tag{14}$$

vhen $\lambda = 0$ and to equation (1) when $\lambda = 1$. The eigenvalues of the matrix A defined by (13) are the roots of he auxiliary equation Ezeilo (2000)

$$r^4 + a_1 r^3 + a_3 r + a_4 = 0 \tag{15}$$

If equation (15) has no root of the form $r = im$, then equation (14) together with the boundary condition (2) has no non trivial solutions, since (7) is satisfied Ezeilo (2000). Therefore the matrix $\left(e^{2\pi A} - I\right)$, ($I$ being the identity matrix) is invertible. Thus, $X = X(t)$ is a 2π periodic solution of (12) if and only if (Hale, 1963)

$$X = \lambda TX, \qquad 0 \le \lambda \le 1 \tag{16}$$

where the transformation T is defined by

$$\left(TX\right)(t) = \int_0^{2\pi} \left(e^{2\pi} - I\right)^{-1} e^{A(t-s)} F\left(X(t), S\right) dt \tag{17}$$

Let S be the space of all continuous 4-vector functions $\bar{X}(t) = \left(x(t),\ y(t),\ z(t),\ u(t)\right)$ which are of period 2π and with norm

$$\left\|\bar{X}\right\|_s = \sup_{0 \le t \le 2\pi} \left\{\left|x(t)\right| + \left|y(t)\right| + \left|z(t)\right| + \left|u(t)\right|\right\} \tag{18}$$

If the operator T defined by (17) is a compact mapping of S into itself then it suffices for the proof of theorem 1 to establish a priori bounds C_7, C_5, C_4, C_{12}, independent of λ such that

$$\left|x\right|_s \le C_7,\ \left|\dot{x}\right|_s \le C_5,\ \left|\ddot{x}\right|_s \le C_4,\ \left|\dddot{x}\right| \le C_{12} \tag{19}$$

see Scheafer (1955)

4. VERIFICATION OF (19)

Let $x(t)$ be a possible 2π periodic solution of equation (10). The main tool to be used here in this verification is the function $V(x, y, z, u)$ defined by

$$V = \tfrac{1}{2}\ddot{x}^2 + \int_0^x g(s)ds + a_4 x\ddot{x} - \tfrac{1}{2}a_4\dot{x}^2 + \ddot{x}h(\dot{x}) \tag{20}$$

The time derivative $\dot{V}$ along the solution path of (11) is

$$\dot{V} = -\dddot{x}f_\lambda(\dddot{x}) + h'_\lambda(\dot{x})\ddot{x}^2 + \ddot{x}\lambda P \tag{21}$$

Integrating (21) with respect to t from $t = 0$ to $t = 2\pi$

$$\int_0^{2\pi} \dot{V}dt = -\int_0^{2\pi} \dddot{x}f_\lambda(\dddot{x})dt + \int_0^{2\pi} h'_\lambda(\dot{x})\ddot{x}^2 dt + \int_0^{2\pi} \ddot{x}\lambda P dt$$

using equation (2), we obtain

$$\int_0^{2\pi} \dddot{x}f(\dddot{x})dt - \int_0^{2\pi} h'(\dot{x})\ddot{x}^2 dt = \int_0^{2\pi} \ddot{x}\lambda P dt$$

$$\int_0^{2\pi} \dddot{x}f(\dddot{x})dt + \int_0^{2\pi} \left|-h'(\dot{x})\right|\ddot{x}^2 dt \le \int_0^{2\pi} \left|\ddot{x}\right|\left|\lambda\right|\left|P\right| dt \tag{22}$$

By (8) and (9), (22) implies

$$\int_0^{2\pi} a_1\dddot{x}^2 dt + \int_0^{2\pi} a_2\ddot{x}^2 dt \le C_1 \int_0^{2\pi} \left|\ddot{x}\right| dt \tag{23}$$

We have used the boundedness of P and the fact that $0 \le \lambda \le 1$ to achieve (23). In particular

$$\int_0^{2\pi} a_1\dddot{x}^2 dt \le C_1 \int_0^{2\pi} \left|\ddot{x}\right| dt$$

or

$$\int_0^{2\pi} \dddot{x}^2 dt \le C_2 \int_0^{2\pi} \left|\ddot{x}\right| dt$$

where $C_2 = a_1^{-1}C_1$, $a_1 \ne 0$

Thus,

$$\int_0^{2\pi} \dddot{x}^2(t)dt \le C_2 \int_0^{2\pi} \left|\ddot{x}\right| dt$$

$$\le C_2(2\pi)^{1/2}\left(\int_0^{2\pi} \ddot{x}(t)dt\right)$$

by Schwartz's inequality. Therefore

$$\left(\int_0^{2\pi} \dddot{x}^2(t)dt\right)^{1/2} \le C_2(2\pi)^{1/2} \equiv C_3 \tag{24}$$

Since $x(0) = x(2\pi)$, there exists $\ddot{x}(\tau_i) = 0$ at some $\tau_i \in \left[0,\ 2\pi\right|$ such that

$$\ddot{x}(t) = \ddot{x}(\tau_1) + \int_{\tau_1}^{t} \dddot{x}(s)ds$$

Then

$$\max_{0\le t\le 2\pi} |\ddot{x}(t)| \le \int_0^{2\pi} |\dddot{x}(t)|dt$$

$$\le (2\pi)^{1/2} \left(\int_0^{2\pi} \dddot{x}(t)dt \right)^{1/2}$$

by Schwartz's inequality. By (24)

$$\max_{0\le t\le 2\pi} |\ddot{x}(t)| \le (2\pi)^{1/2} C_3 \equiv C_4$$

Therefore

$$|\ddot{x}|_\tau \le C_4 \qquad (25)$$

Also since $x(0) = x(2\pi)$ by (2), there exist $\dot{x}(\tau_2) = 0$ at some $\tau_2 \in [0,\ 2\pi]$ such that

$$\dot{x}(t) = \dot{x}(\tau_2) + \int_{\tau_2}^{t} \ddot{x}(s)ds$$

so that

$$\max_{0\le t\le 2\pi} |\dot{x}(t)| \le \int_0^{2\pi} |\ddot{x}(t)|dt$$

$$\le (2\pi)^{1/2} \left(\int_0^{2\pi} \ddot{x}(t)dt \right)^{1/2}$$

by Schwartz's inequality. In view of (25), we have

$$|\dot{x}|_\tau \le C \qquad (26)$$

Now integrating (10) with respect to *t* from *t* = *0* to *t* = 2*π* and using (2) yields

$$\int_0^{2\pi} a_4 x dt = \int_0^{2\pi} \lambda P dt - \int_0^{2\pi} f_\lambda(\dddot{x})dt - \int_0^{2\pi} \lambda g(\ddot{x})dt - \int_0^{2\pi} h_\lambda(\dot{x})dt \qquad (27)$$

With bounds on $\dddot{x}, \ddot{x}, \dot{x}$ in (24), (25) and (26) respectively and the boundedness of P as specified in (iii) of the hypotheses of theorem 1, the right hand side of (27) is bounded.
That is

$$\int_0^{2\pi} |\lambda P|dt + \int_0^{2\pi} |f_\lambda(\dddot{x})|dt + \int_0^{2\pi} |\lambda g(\ddot{x})|dt + \int_0^{2\pi} h_\lambda(\dot{x})dt \le C_6$$

Therefore

$$\int_0^{2\pi} a_4 x dt \le C_6$$

which implies that

$$\int^{2\pi} x dt \le C \qquad (28)$$

where $C = a_4^{-1} C_6,\ a_4 \ne 0$

From section 2

$$|x|_1 = \int_0^{2\pi} x dt$$

That is

$$|x| \le C$$

Also,

$$\max |x(t)| \le \int_0^{2\pi} |x(t)dt|$$

implies

$$\max |x(t)| \le C_7$$

$$|x|_\tau \le C \qquad (29)$$

Now it remains the fourth inequality in (19) for our theorem 1 to be fully verified. Note that equation (10) can be expressed in the form

$$x^{(4)} + f_\lambda(\ddot{x}) = \eta_0 \quad (30)$$

where

$$\eta_0 = \lambda P - \lambda g(\ddot{x}) - h_\lambda(\dot{x}) - a_4 x$$

with bounds on x, $\dot{x}$, $\ddot{x}$ in (29), (26) and (25) respectively together with the bounded of P and the fact that $0 \le \lambda \le 1$, then

$$|\eta_0| \le C_8 \quad (31)$$

Therefore

$$x^{(4)} + f_\lambda(\ddot{x}) \le C_{10} \quad (32)$$

Multiplying (32) by $x^{(4)}$ and integrating with respect to t from $t = 0$ to $t = 2\pi$ yields

$$\int_0^{2\pi} x^{(4)^2}(t)dt + \int_0^{2\pi} f_\lambda(\ddot{x})x^{(4)}dt \le |\eta_0| \int_0^{2\pi} |x^{(4)}|dt$$

Since f is a continuous function and f is defined as in section 2, there are constants C_η, C_{12} such that

$$|x^{(4)}|_2^2 \le C_\eta |x^{(4)}|_2 + C_{10} |x^{(4)}|_2 \quad (33)$$

Hence

$$|x^{(4)}|_2 \le C_{11} \quad (34)$$

from which because of (2) with $r = 3$ then

$$|\dddot{x}|_2 \le C_{12} \quad (35)$$

CONCLUSION

The estimates (25), (26), (29) and (35) verify the inequality (19) and hence the proof theorem 1, which implies existence of Periodic Solutions for equations (1) – (2).

REFERENCES

Ezeilo, J. O. C., 1979. Periodic solutions of a certain fourth order differential equation, Atti Acad, Nazi, Lincei CL Sci Fis Mat Natur vol LXVII 344-350.

Ezeilo, J. O. C., 1999. Non resonant oscillations for some fourth order differential equations I. Directional in mathematics 15-25.

Ezeilo, J. O. C., 2000. Periodic BVP for some fourth order differential equations, Proc. Of National Mathematics Centre, Abuja, Nigeria, Vol 1 Number 1, 1-14.

Ezeilo, J. O. C and Tejumola, H. O., 2001. Non resonant oscillations for some fourth order differential equation II. Journal of Nigerian Mathematical Society, Vol 20, 45-58.

Hale, T., 1963 Oscillations in nonlinear systems, McGraw-hill, New York, Toronto, London.

Leray J and Schauder, J., 1934. Topologie et equations functionelles, Ann. Sci Ecele norm sup (3) 51, 45-78.

Ogbu, H. M., 2006. Existence of periodic solutions for a boundary value problem of a fourth order differential equation. Journal of Information, Communication and Computing Technologies, Ebonyi State University, Abakaliki – Nigeria. Vol. 2, 85-88

Ogbu, H. M., 2007 Existence of periodic solutions for a certain boundary value problem of a non linear fourth order differential equation, Pacific Journal of Science and Technology USA 8(1) 130-136

Scheafer, H., 1955. Uber der methode de a priori schranken, math ann 129 415 – 416

Tejumola, H. O., 2006. Periodic boundary value Problems of some fifth, fourth and third ordinary differential equations, J. of Nigerian Math. Soc. Vol. 25 pp 37-46

Tiryaki, A., 1990. On periodic solutions of certain fourth and fifth order differential equation, Pure and Applied Mathematical Sciences Vol. xxxii Number 1-2, 15-19