

DISCRETE TIME MODEL OF HIV TRANSMISSION AND THE IMPACT OF AIDS ON ECONOMIC PRODUCTIVITY

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ABSTRACT

The objective of this work is to formulate a discrete time deterministic model of HIV transmission and the impact of AIDS on economic productivity of a population (Nigeria). The population is divided into eight segments on epidemiological basis. For the economics model, we use the Cobb-Douglas production function. The effect of AIDS on capital stock and labour input in a given period form the economic aspect of the model. The model is analyzed using techniques from Dynamical Systems Theory. The steady states are determined and sufficient conditions for their stability established. From the study, it can be concluded that the AIDS pandemic has effect on the depreciation of stock of capital at steady state. In addition, there is no state in which there are no susceptible.

KEYWORDS: Discrete time model, Impact of AIDS, steady state, dynamical systems theory, depreciation of stock.

1. INTRODUCTION

Hyman, et al, 1999, presented a model referred to as differential infectivity (DI) HIV model that accounts only for differences between infected individuals. The infected population was subdivided into subgroups. In this DI model, it was assumed that individuals enter a specific group when they become infected and stay in that group until they are no longer involved in transmitting the disease. Their infectivity and progression rates to AIDS were assumed to depend upon which group they were in. The susceptible population was assumed to be homogeneous, and variation in susceptibility, risk neglected. They investigated the steady state and its stability.

Hyman, and Li, 2000 used the simple DI model formulated in Hyman et al 1999 and generalized it to a model in which the risk level of infected individuals depended on the group to which they belong. They demonstrated how a mean number of contacts, a mean transmission probability and a mean duration of infection could be defined. They also obtained the reproductive number for the model.

Hyman, et al. 1999 presented an SP (staged progression) model with a homogeneous contact rate. It assumed that the susceptible population is homogeneous and is maintained by the same inflow. It also assumed that the population of infected individuals is subdivided into n subgroups with different infection stages such that the infected individuals enter the first subgroup 1_1 and then gradually progress from this subgroup to subgroup 1_n . they obtained the mean duration of infection, the mean probability of transmission and the reproductive number.

Hyman, et al. 1999 presented a simple differential susceptibility (DS) model in which the infected population is homogeneous, but the susceptible population is divided into n groups according to their susceptibilities. They also combined the DS and DI models. In each of these models, they obtained the reproductive number. They finally developed the segregated risk DI model and simple age structured model. In each of the models, they obtained the reproductive number.

A lot of work has been done on the impact of HIV/AIDS on socio-economic activities though none has been reported for Nigeria.

Stover and Stannecki 2003 reported the results of research and analysis undertaken by U.S. Census Bureau staff through the UNAIDS Reference Group. In the review, the reference group has

- Developed a new model for estimating and projecting HIV prevalence from available surveillance data. This model called the Epidemic projection package (EPP) replaces Episode used in previous estimates.
- Recommended the use of the spectrum software to make the demographic and HIV projections based on the prevalence estimate
- Reviewed the latest and revised previous assumptions about the distribution of HIV infection to AIDS death.

- Developed new procedures for estimating paternal and double AIDS orphans.

Stover, and Stanek. 2003 reported studies on the economic impacts of AIDS. They reported that a World Bank study of the economic impacts of AIDS in Africa concluded that macro-economic impacts of AIDS could be significant. He went further to report other studies as follows:

A simulation model of the economy of Cameroon concluded that the annual growth rate of GDP could have been reduced by as much as 2% points during the 1987-1990 period because of AIDS. Hyman, et al. 1999.

A study of the macroeconomic impact of AIDS in Zambia found that by 2000 the GDP would be 5 to 10% lower because of AIDS than it would be if there were no AIDS affecting the population. The authors concluded "without unprecedented infusions of free foreign aid to mitigate the effects of AIDS, the economy of Zambia will suffer considerable damage. Bollinger, and Stover. 1999.

In an assessment of the macroeconomic impact of AIDS in Tanzania in 1991 by the government of Tanzania, the World Bank and the World Health Organization found that total GDP will decrease by 15 to 25% smaller in 2010 because of the impact of AIDS, Cuddington 1993.

A study of the impact of AIDS on the economy of Kenya projected that GDP will be 14% lower in 2005 than it would have been without AIDS and that GDP per capita will be 10% less in 2005 than it was in 1999, John, et al. 1996.

Nicholls, 2002 modeled the macroeconomic impact of HIV/AIDS in the English speaking Caribbean. They developed a model known as CARIBAIDEMOD which was based on a modification of the economic model developed by Cuddington 1993, Cuddington and Hancock 1994 and Cuddington, Hancock and Rogers 1994.

Cuddington, 1993 modeled the macroeconomic effects of AIDS with application to Tanzania. He developed a simple model for analyzing the effects of AIDS on the growth paths of potential GDP and per capita GDP. The classic Slow (1956) growth model was extended to incorporate the key macroeconomic consequences of AIDS. Using this model, he compared AIDS and non-AIDS scenarios analytically and by simulation based on Tanzanian demographic and macroeconomic data. The model was used to discuss the likely effect on the ratio of capital to labour and on output per capita as the economy moves from a non-AIDS situation towards a new steady state, in which AIDS is assumed to be endemic.

Arndt and Lewis 2000, reported on the preliminary results from an analysis of the macro impact of HIV/AIDS in South Africa. They constructed an economy-side simulation model that embodies the important structural features of the South Africa economy and the major impact channels of the HIV/AIDS epidemic. Together with available demographic estimates for their part of the epidemic (on labour supply, death rates, and HIV prevalence) and the assumptions about behavioural and policy responses (household and government spending on health, slower productivity growth), they used the model to generate and compare two scenarios: a hypothetical non-AIDS scenario in which the economy continues to perform as it had been doing during the last several years and an AIDS scenario in which the key AIDS related factors affect economic performances. Over the 1997 – 2010 simulation period, GDP growth rates in the two scenarios diverge steadily, reaching a maximum differential of 2.6% points. This results in a GDP level in 2010 that is 17% lower in the AIDS scenario, an alternative measure of non-health, non-food absorption is 21% lower by 1020. While some of this decline is due to the lower population associated with the AIDS scenario per capita GDP does not drop by around 7%.

Bollinger, et al. 2000 worked on the economic impact of AIDS in Malawi. They reported researches conducted in Malawi saying "recent projections indicate that the annual number of new AIDS cases will double from 64,260 in 1998 to 99,400 in 2005, reaching 136,350 each year by 2015. The number of pediatric AIDS cases each year is projected to rise from 14,500 in 1998 to 21,240 in 2005, reaching 25,510 by the year 2015. The projections also show that the number of maternal orphans in Malawi will increase from about 250,000 in 1998 to 660,000 by the year 2005 and reach over one million by 2015".

According to Cohen, 1997, the infant mortality rate in Namibia will be 40% higher by 2010 than it was in 1997 due to the impact of AIDS, the child mortality rate will double, and life expectancy will decrease from 56.8 to 29.5. Overall population growth will be reduced from 2.2% to 0.1% because of the effect of AIDS.

Burger, 2000, studied the macroeconomic impact of HIV/AIDS in South Africa. The study reported the result from a supply-side analysis of the macroeconomic impact of HIV/AIDS in South Africa. The theoretical model used was a disaggregated Cuddington-type model. The economic model predicted that economic growth will decline rapidly over the next 15 years. By examining the differential between the AIDS scenario and the non-AIDS scenario, it appears as if the impact of the epidemic is likely to be substantial. In this model, the labour force is the more important channel through which HIV/AIDS operates as opposed to the alternative impact on capital formation. Finally, in this model, the population impact of HIV/AIDS on the population exceeds the effects of the pandemic on GDP leading to a projected rise in per capita GDP in agreement with previous studies by Over, 1992.

2. FORMULATION OF THE MODEL

To construct the model, we divide the entire Nigerian population into eight segments each corresponding to one of the epidemiological state, which are P(population of those between 15 years and 65 years), N(population of those below 15 years), S(uninfected susceptible group mainly adults above 15 years), 1_1 (incubation state 1), 1_2 (incubation state 2), 1_3 (incubation state 3), A(final stage of fully developed AIDS) and R (resistant group who have heeded to the campaigns against HIV/ AIDS). Each of the S, 1_1 , 1_2 , 1_3 , A segments is subdivided into three risk groups (heterosexuals, intravenous drug users (IDU) and homosexual), and each of these risk groups is further subdivided into three subgroups on rate of acquisition of sexual partners (low, medium, high).

When those in N class reach 15 years, they may join R group or S group depending on the level of awareness they have on HIV/AIDS. Some people who are already in R (the reserved or resistant group) may later change their minds and start living care free lives again thereby joining S class. Immediately some one in S class gets infected, he joins group 1_1 . After some time, he graduates into 1_2 class and later enters 1_3 group. After some time in 1_3 (which may vary from person to person), he develops full blow AIDS, thereby joining A class. This description is as shown in the flow diagram below (Fig. 2.1).

The problem now is to develop a discrete time deterministic model on the transmission dynamics of HIV in this setting.

2.1 Flow Diagram

Below is a flow diagram representing the model.

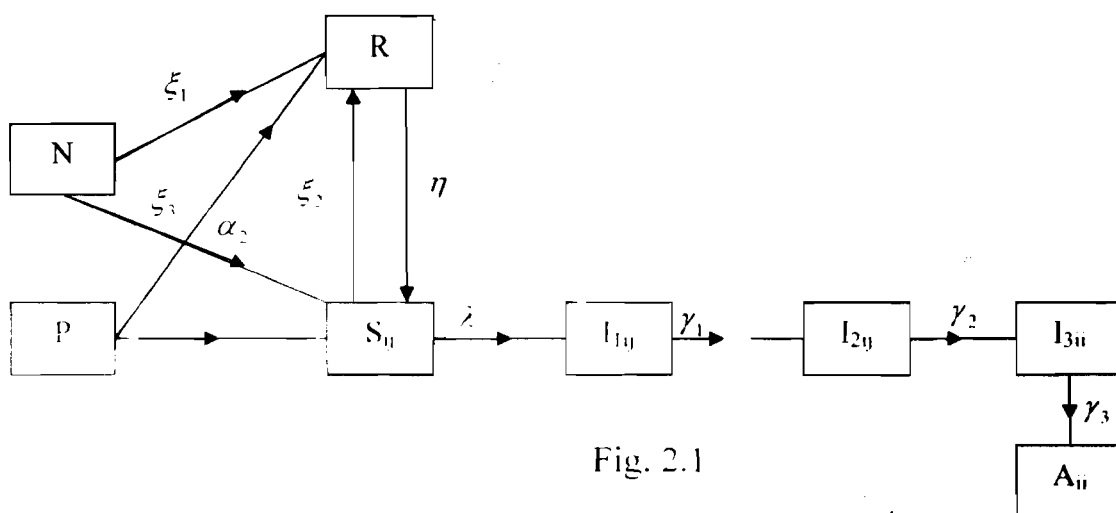


Fig. 2.1

From this flow diagram we observe that some susceptibles may heed the campaigns against HIV/AIDS and withdraw to the resistant group while some members of the resistant group may later become exposed to HIV infection (susceptibility).

2.2 Assumptions

To construct this model, we present the following assumptions:

1. There is no recovered class.
2. There are only two ways by which susceptibles may become infected: through sexual contact or by the sharing of needles, IDUS (intravenous drug users) and other sharp objects like razor blades, barbing materials, etc. Material transmission and transmission via blood products are classified under sharing of needles.
3. There are varying rates of passage through each incubation stage
4. Individuals in the AIDS compartment are neither sexually active nor do they share needles, and they leave the population solely by AIDS related mortality.
5. The proportion of new recruits allocated to each compartment is held constant over a time period.
6. Heterosexuals, IDU's and homosexuals have the same rate of acquisition of sexual partners, that is $J = 1$.
7. Those below 15 years are equally likely to join any of the i groups (rate of acquisition of sexual partners).

8. ξ_1, ξ_2, ξ_3 are time independent or time homogeneous.
9. ρP is closed to migration (that is, there is no migration into P).
10. ξ_1, ξ_2, ξ_3 and η are dependent on the rate of death as a result of AIDS (δ). The more δ , the more ξ_1, ξ_2, ξ_3 but the less η .
11. N is a function of P and Y_t .

2.3 SYMBOLS AND PARAMETERS

- i = risk group (heterosexuals, IDU's, homosexuals), $i = 1, 2, 3$.
 j = rate of acquisition of sexual partners (low, medium, high), $j = 1, 2, 3$
 R = Resistant group – those that may withdraw from sexual activities due to self-discipline or due to anti AIDS campaigns.
 S = Susceptible group.
 I_1 = Those in incubation state 1. Initial period of HIV antigen concentration shortly after infection.
 I_2 = Those in incubation stage 2. The longer asymptomatic stage.
 I_3 = Those in incubation stage 3. Final incubation stage with high level of antigenaemia.
 A = The state of fully developed AIDS.
 δ = AIDS related mortality rate.
 γ_k = The average rate of progression from incubation stage k to incubation stage $k + 1$, $k = 1, 2, 3$.
 μ = Non-AIDS or natural mortality rate.
 λ = Infection rate or per capita force of infection
 η = $\beta_0 \exp(-\epsilon\delta)$ = transition rate from the reservoir (resistant group) to the susceptible group.
 ξ_1 = $\beta_0 \exp(-\epsilon_1\delta)$ = transition rate of those below 15 years who graduate into 15 years and move to the resistant group (reservoir).
 ξ_2 = $\beta_0 \exp(-\epsilon_2\delta)$ = transition rate of susceptible class to the resistant (reservoir group)
 ξ_3 = $\beta_0 \exp(-\epsilon_3\delta)$ = transition rate of those aged between 15 and 65 years to the resistant (reservoir) group.
 β' = Birth rate of the population of those between 15 and 65 years
 α_1 = Per capita rate at which the population of age 15 – 65 years join the susceptible class
 α_2 = Per capita rate at which the population aged below 15 years join the susceptible class
 v = $\alpha_2 + \xi_1$
 P = The population size of those between the age of 15 and 65 years
 N = The population size of those below 15 years. N is a function of P and Y_t , that is $N = f(P, Y_t) = \mu_1 P^\phi Y_t^\phi$, $0 < \phi < 1$ and $(f(0, Y_t)) = f(P, 0) = 0$
 β_0 = A parameter.

$$P = \sum_{i=1}^3 \sum_{j=1}^3 (S_{ij} + I_{1ij} + I_{2ij} + I_{3ij} + A_{ij}) + R$$

$$\Lambda = \alpha_1 P + \alpha_2 N + \eta R$$

$$= \alpha_1 \left[\sum_{i=1}^3 \sum_{j=1}^3 (S_{ij} + I_{1ij} + I_{2ij} + I_{3ij} + A_{ij}) + R \right] + \alpha_2 N + \eta R$$

By assumption (4), A_{ij} does not contribute in Λ . Hence,

$$\Lambda = \alpha_1 \left[\sum_{i=1}^3 \sum_{j=1}^3 (S_{ij} + I_{1ij} + I_{2ij} + I_{3ij}) + R \right] + \alpha_2 N + \eta R$$

2.4 THE EPIDEMIOLOGICAL MODEL (CONTINUOUS TIME VERSION)

Using the above flow diagram, symbols and parameters, the model is presented as

$$\begin{aligned}
 \frac{dS_{ij}}{dt} &= \Lambda - S_{ij}(\mu + \lambda_{ij} + \xi_2) \\
 \frac{dI_{1ij}}{dt} &= S_{ij}\lambda_{ij} - I_{1ij}(\mu + \gamma_{1ij}) \\
 \frac{dI_{2ij}}{dt} &= I_{1ij}\gamma_{1ij} - I_{2ij}(\mu + \gamma_{2ij}) \\
 \frac{dI_{3ij}}{dt} &= I_{2ij}\gamma_{2ij} - I_{3ij}(\mu + \gamma_{3ij}) \\
 \frac{dA_{ij}}{dt} &= I_{3ij}\gamma_{3ij} - \delta A_{ij} \\
 \frac{dR}{dt} &= \xi_1 N + \xi_2 S_{ij} + \xi_3 P - (\eta + \mu)R \\
 \frac{dN}{dt} &= \beta' P - (\mu + \nu)N, \nu = \alpha_2 + \xi_1
 \end{aligned} \tag{2.1}$$

2.5 DERIVATION OF THE DISCRETE TIME VERSION OF THE EPIDEMIOLOGICAL MODEL

Here we adopt a method used by Sowunmi, 1993, as follows: Consider

$$\frac{dy}{dt} = \lambda y + f \tag{2.2}$$

Let us consider $\frac{dy}{dt} = \lambda y$ i.e. $f = 0$

$$\frac{dy}{y} = \lambda(t)dt$$

$$\ln y = \int_0^t \lambda(s)ds + c$$

$$\begin{aligned}
 y(t) &= \exp\left\{\int_0^t \lambda(s)ds\right\} \cdot e^c \\
 &= y(0) \exp\left\{\int_0^t \lambda(s)ds\right\} = y(0)\pi(0,t)
 \end{aligned}$$

where

$$\pi(0,t) = \exp\left\{\int_0^t \lambda(s)ds\right\} = \pi$$

Let

$$y(t) = A(t)\pi(0,t) \tag{2.3}$$

$$\begin{aligned}
 \frac{dy}{dt} &= \pi \frac{dA}{dt} + A(t) \frac{d\pi}{dt} = \lambda y + f \\
 &= \lambda A\pi + f
 \end{aligned}$$

$$\pi \frac{dA}{dt} + A(t)\lambda(t)\pi = \lambda A\pi + f$$

$$\frac{dA}{dt} = \pi^{-1} f(t) = f(t) \exp\left(-\int_0^t \lambda(s)ds\right)$$

$$A(t) = y(0) + \int_0^t f(\tau) \exp\left(-\int_0^\tau \lambda(s)ds\right) d\tau \tag{2.4}$$

Put (2.4) in (2.3) to get,

$$y(t) = y(0)\pi(0, t) + \int_0^t \left[f(\tau) \exp\left(-\int_0^\tau \lambda(s)ds\right) \left(\exp \int_0^t \lambda(s)ds \right) \right] d\tau$$

$$= y(0)\pi(0, t) + \int_0^t f(\tau) \exp\left(\int_0^\tau \lambda(s)ds\right) d\tau$$

$$y(t) = y(0)\pi(0, t) + \int_0^t f(\tau)\pi(\tau, t)d\tau$$

$$y(t+1) = y(0)\pi(0, t+1) + \int_0^{t+1} f(\tau)\pi(\tau, t+1)d\tau$$

N.B. $\pi(0, t+1) = \pi(0, t)\pi(t, t+1)$

$$y(t+1) = y(0)\pi(0, t)\pi(t, t+1) + \int_0^t f(\tau)\pi(\tau, t+1)d\tau + \int_t^{t+1} f(\tau)\pi(\tau, t+1)d\tau$$

$$y(t+1) = \pi(t, t+1) \left[y(0)\pi(0, t) + \int_0^t f(\tau)\pi(\tau, t)d\tau \right] + \int_0^1 f(t+\tau)\pi(\tau+t, t+1)d\tau$$

$$= \pi(t, t+1)y(t) + \int_0^1 f(t+\tau)\pi(\tau+t, t+1)d\tau$$

At $\tau = 0$

$$y(t+1) = \pi(t, t+1) y(t) + f(t)\pi(t, t+1)$$

$$\Rightarrow y(t+1) = \pi(t, t+1) [y(t) + f(t)]$$

(2.5)

where

$$\pi(t, t+1) = \exp\left(\int_t^{t+1} \lambda(s)ds\right)$$

(2.6)

is the transition probability.

Applying (2.5) to our model in (2.1) we have the discrete version of (2.1) as

$$\left. \begin{aligned} S_{j,t+1} &= (S_{j,t} + \Lambda_t) \pi_s \\ I_{1j,t+1} &= (I_{1j,t} + S_{j,t} \lambda_{1j,t}) \pi_1 \\ I_{2j,t+1} &= (I_{2j,t} + I_{1j,t} \gamma_{1j}) \pi_2 \\ I_{3j,t+1} &= (I_{3j,t} + I_{2j,t} \gamma_{2j}) \pi_3 \\ A_{j,t+1} &= (A_{j,t} + I_{3j,t} \gamma_{3j}) \pi_A \\ R_{t+1} &= (R_t + \xi_1 N_t + \xi_2 S_{j,t} + \xi_3 P_t) \pi_R \\ N_{t+1} &= (N_t + B' P_t) \pi \end{aligned} \right\} \quad (2.7)$$

2.7 THE ECONOMIC MODEL

2.7.1 Parameters and Symbols

Y_t = Aggregate output (production) in period t or total output.

L_{nt} = The labour force of age n in period t.

ρ_{nt} = Productivity of labourers of age n who are AIDS free at period t.

ρ'_n = Productivity of labourers of age n who have AIDS at period t.

K_t = Capital stock in period t.

β = Share of national output accruing to labour.

γ = Rate of technological change or total factor of productivity over time.

E_t = Labour input in period t (measured in units of effective labour) or labour input.

λ = Employment rate.

α = A constant factor.

e = Proportion of work time (in years) lost due to AIDS per AIDS patient

a_{nt} = Proportion of the population of age n that is suffering from AIDS in period t.

m = Total savings rate in the no AIDS scenario.

m' = Total savings rate in the AIDS scenario.

H_t = Total annual cost of treating AIDS.

f = A fraction of the total cost of AIDS financed out of savings.

r = Rate of depreciation of stock of capital.

2.7.2 Assumptions:

- (1) Labour force in Nigeria is aged between 15 and 65 years.
- (2) Productivity varies only with age.
- (3) H_t is arbitrarily defined, it depends on the government

2.7.3 The Model

The Cobb – Douglas production function(has contain in Cuddington, 1993) is given as

$$Y_t = \alpha \gamma^t E_t^\beta K_t^{1-\beta} \quad (2.8)$$

where

$$E_t = \rho_t L_t$$

The two factors affected by HIV/AIDS in this model are K_t and E_t .

Effect on K_t

Total savings for the previous years,

$$S_t = mY_t - fH_t \quad (2.9)$$

Current value of capital stock of the previous years

= the value of the capital stock of the previous years – depreciation

$$\Rightarrow K_t = (1 - r)K_{t-1} \quad (2.10)$$

Current stock of capital = Total savings during the past year + current value of the capital stock of the previous year.

$$\Rightarrow K_t = S_{t-1} + K_{t-1}$$

Using (2.9) and (2.10) we have

$$K'_t = mY_{t-1} + (1 - r) K_{t-1} - fH_{t-1} \quad (2.11)$$

$$\Rightarrow K'_{t+1} = mY_t + (1 - r) K_t - fH_t \quad (2.12)$$

But

$$m' = \frac{mY_t - fH_t}{Y_t} \Rightarrow m'Y_t = mY_t - fH_t$$

$$\therefore K'_{t+1} = m'Y_t + (1 - r) K_t$$

Effect on E_t

The Cobb – Douglas production function given in (2.8) assumes that the labour force has full employment, but in Nigeria, unemployment is the order of the day. Hence we bring in an “employment rate” parameter, λ . Hence

$$E_t = \lambda \rho_t L_t \quad (2.13)$$

Bringing in the effect of AIDS in (2.13), the labour productivity term is now modified to

$$E'_t = \sum_{n=15}^{65} [(1 - ea_m) \lambda_n \rho'_m L_m] \quad (2.14)$$

Hence, our Economic model becomes

$$\begin{aligned} K'_{t+1} &= m\alpha \gamma^t E_t^\beta K_t^{1-\beta} + (1-r)K_t - fH_t \\ &= mY_t + (1-r) K_t - fH_t \\ &= m'Y_t + (1-r) K_t \end{aligned} \quad (2.15)$$

$$Y_t = \alpha \gamma^t E_t^\beta K_t^{1-\beta}$$

$$L_t = \sum_{i=1}^3 \sum_{j=1}^3 (S_{ij}(n) + I_{1ij}(n) + I_{2ij}(n) + I_{3ij}(n)) + R_t(n)$$

$$E'_t = \sum_{n=15}^{65} [(1 - ea_m) \lambda_n \rho'_m L_m]$$

$$m'Y_t = mY_t - fH_t$$

2.8 THE MODEL

Adding (2.15) to (2.7) we obtain the model below which is a combination of the epidemiological model and economic model. This is the model we are going to investigate

$$\begin{aligned} S_{ij,t+1} &= \pi_s S_{ij,t} + \wedge_t \pi_s &= (S_{ij,t} + \wedge_t) \pi_s \\ I_{1ij,t+1} &= \pi_1 I_{ij,t} + \lambda_{ij,t} \pi_1 S_{ij,t} &= (I_{1ij,t} + S_{ij,t} \lambda_{ij,t}) \pi_1 \\ I_{2ij,t+1} &= \pi_2 I_{2ij,t} + I_{1ij,t} \gamma_{1ij,t} \pi_2 &= (I_{2ij,t} + I_{1ij,t} \gamma_{1ij,t}) \pi_2 \end{aligned} \quad (2.16)$$

$$\begin{aligned}
 I_{3ij,t+1} &= \pi_3 I_{3ij,t} + I_{2ij,t} \gamma_{2ij} \pi_3 &= (I_{3ij,t} + I_{2ij,t} \gamma_{2ij}) \pi_3 \\
 A_{ij,t+1} &= \pi_A A_{ij,t} + I_{3ij,t} \gamma_{3ij} \pi_A &= (A_{ij,t} + I_{3ij,t} \gamma_{3ij}) \pi_A \\
 R_{t+1} &= (R_t + \xi_1 N_t + \xi_2 S_{ij,t} + \xi_3 P_t) \pi_R &= (R_t + \xi_1 N_t + \xi_2 S_{ij,t} + \xi_3 P_t) \pi_R \\
 N_{t+1} &= \pi N_t + \beta' P_t \pi &= (N_t + \beta' P_t) \pi \\
 K'_{t+1} &= m \alpha \gamma' E_t^\beta K_t^{1-\beta} + (1-r)K_t - fH_t \\
 &= m' Y_t + (1-r)K_t
 \end{aligned}$$

where

$$\begin{aligned}
 Y_t &= \alpha \gamma' E_t^\beta K_t^{1-\beta} \\
 E_t' &= \sum_{n=1}^6 [(1 - e a_n) \lambda_n \rho_n' L_n] \\
 L_t &= \sum_{i=1}^3 \sum_{j=1}^3 (S_{ijt} + I_{1ijt} + I_{2ijt} + I_{3ijt}) + R_t
 \end{aligned}$$

Applying assumption 6, (2.16) reduces to

$$\left. \begin{aligned}
 S_{i,t+1} &= (S_{i,t} + \dots) \pi_s \\
 I_{1i,t+1} &= (I_{1i,t} + S_{i,t} \lambda_{1i}) \pi_1 \\
 I_{2i,t+1} &= (I_{2i,t} + I_{1i,t} \gamma_{1i}) \pi_2 \\
 I_{3i,t+1} &= (I_{3i,t} + I_{2i,t} \gamma_{2i}) \pi_3 \\
 A_{i,t+1} &= (A_{i,t} + I_{3i,t} \gamma_{3i}) \pi_A \\
 R_{t+1} &= (R_t + \xi_1 N_t + \xi_2 S_{i,t} + \xi_3 P_t) \pi_R \\
 N_{t+1} &= (N_t + \beta' P_t) \pi \\
 K'_{t+1} &= m' Y_t + (1-r)K_t
 \end{aligned} \right\} \tag{2.17}$$

where

$\pi_s, \pi_1, \pi_2, \pi_3, \pi_A, \pi_R, \pi$ are transition probabilities from one state to another.

3. STEADY STATE AND ITS STABILITY

3.1 Steady State

Proposition 1

There exists one and only one non-trivial steady state satisfying equation (2.17).

The steady state is given as

$$(S_t^0, I_{1t}^0, I_{2t}^0, I_{3t}^0, A_t^0, R_t^0, N_t^0, K_t^0) \tag{2.18}$$

where

$$S_t^0 = U = \frac{[b + \eta(\xi_1 \beta' a_s + \dots) - k] - b \beta' a_s}{1 - \eta \xi_2 a_R a_s}$$

$$I_{1t}^0 = U \lambda_{1i} a_1$$

$$I_{2t}^0 = U \gamma_{1i} \lambda_{1i} a_1 a_2$$

$$I_{3t}^0 = U \gamma_{1i} \gamma_{2i} \lambda_{1i} a_1 a_2 a_3$$

$$A_t^0 = U \gamma_{1i} \gamma_{2i} \gamma_{3i} \lambda_{1i} a_1 a_2 a_3 a_A$$

$$R_t^0 = \frac{[\alpha_1 \xi_2 a_s + \alpha_2 \xi_2 \beta' a_s + \xi_1 \beta' a + \xi_3] a_R P}{1 - \eta \xi_2 a_R a_s}$$

$$N_t^0 = \beta' P a$$

$$K_t^0 = \left(\frac{m' \alpha \gamma' E^\beta}{r} \right)^{1/\beta}$$

$$\alpha_r = \frac{\pi_r}{1 - \pi_r}, r = 1, 2, 3, A, R, S$$

Proof

At steady state

$$S_{i,t+1} = S_{i,t} = S_i, I_{ki,t+1} = I_{ki,t} = I_{ki}, k = 1, 2, 3$$

$$A_{i,t+1} = A_{i,t} = A_i, R_{t+1} = R_t = R$$

$$N_{t+1} = N_t = N, K_{t+1} = K_t = K$$

This implies that

$$S_t = \pi_S S_t + \Delta \pi_S \tag{i}$$

$$I_{1t} = \pi_1 I_{1t} + S_t \lambda_1 \pi_1 \tag{ii}$$

$$I_{2t} = \pi_2 I_{2t} + I_{1t} \gamma_1 \pi_2 \tag{iii}$$

$$I_{3t} = \pi_3 I_{3t} + I_{2t} \gamma_2 \pi_3 \tag{iv}$$

$$A_t = \pi_A A_t + I_{3t} \gamma_3 \pi_A \tag{v}$$

$$R = \pi_R R + (\xi_1 N + \xi_2 P) \pi_R \tag{vi}$$

$$N = \pi N + \beta' P \pi \tag{vii}$$

$$K = m' \alpha \gamma' E^\beta K^{1-\beta} + (1-r) K \tag{viii}$$

From (i)

$$S_t = \Delta \frac{\pi_S}{1 - \pi_S} = \Delta a_S, \text{ where } a_S = \frac{\pi_S}{1 - \pi_S}$$

But

$$\Delta = \alpha_1 P + \alpha_2 N + \eta R$$

$$S_t = (\alpha_1 P + \alpha_2 N + \eta R) a_S \tag{1}$$

Similarly from (ii) to (viii)

$$I_{1t} = \lambda_1 \left(\frac{\pi_1}{1 - \pi_1} \right) S_t = \lambda_1 a_1 S_t, \text{ where } a_1 = \frac{\pi_1}{1 - \pi_1} \tag{2}$$

$$I_{2t} = \gamma_1 I_{1t} \left(\frac{\pi_2}{1 - \pi_2} \right) = \gamma_1 \lambda_1 a_1 a_2 S_t, \text{ where } a_2 = \frac{\pi_2}{1 - \pi_2} \tag{3}$$

$$I_{3t} = \gamma_2 I_{2t} \left(\frac{\pi_3}{1 - \pi_3} \right) = \gamma_1 \gamma_2 \lambda_1 a_1 a_2 a_3 S_t, \text{ where } a_3 = \frac{\pi_3}{1 - \pi_3} \tag{4}$$

$$A_t = \gamma_3 I_{3t} \left(\frac{\pi_A}{1 - \pi_A} \right) = \gamma_1 \gamma_2 \gamma_3 \lambda_1 a_1 a_2 a_3 a_A S_t, \text{ where } a_A = \frac{\pi_A}{1 - \pi_A} \tag{5}$$

$$R = (\xi_1 N + \xi_2 S_t + \xi_3 P) \frac{\pi_R}{1 - \pi_R} = (\xi_1 N + \xi_2 S_t + \xi_3 P) a_R,$$

where

$$a_R = \frac{\pi_R}{1 - \pi_R} \tag{6}$$

$$N = \beta' P \left(\frac{\pi}{1 - \pi} \right) = \beta' P a, \text{ where } a = \frac{\pi}{1 - \pi} \tag{7}$$

$$K = \left(\frac{m \alpha \gamma' E^\beta}{r} \right)^\beta \tag{8}$$

Putting (6) and (7) in (1) to solve for S_t and then solve $I_{1t}, I_{2t}, I_{3t}, A_t$ and R , we have the steady state as:

$$(S_t^0, I_{1t}^0, I_{2t}^0, I_{3t}^0, A_t^0, R^0, N^0, K^0)$$

where

$$S_t^0, I_{1t}^0, I_{2t}^0, I_{3t}^0, A_t^0, R^0, N^0, K^0$$

are as defined above

Linearizing (2.17) we obtain the matrix, D

3.2 STABILITY

The characteristic equation of our matrix D is

$$|D - \lambda I| = [(1 + \alpha_1)\pi_S - \lambda]^3 (\pi_1 - \lambda)^3 (\pi_2 - \lambda)^3 (\pi_3 - \lambda)^3 (\pi_A - \lambda)^3 [(\xi_3\pi_R - \lambda)(\pi - \lambda) - \beta'\pi\xi_1\pi_R] (\lambda - \lambda) = 0$$

Solving gives

$$\lambda_1 = \lambda_2 = \lambda_3 = (1 + \alpha_1)\pi,$$

$$\lambda_4 = \lambda_5 = \lambda_6 = \pi$$

$$\lambda_7 = \lambda_8 = \lambda_9 = \pi_2$$

$$\lambda_{10} = \lambda_{11} = \lambda_{12} = \pi_3$$

$$\lambda_{13} = \lambda_{14} = \lambda_{15} = \pi_A$$

$$(\xi_3\pi_R - \lambda)(\pi - \lambda) - \beta'\xi_1\pi_R\pi = 0 \Rightarrow \lambda^2 - (\xi_3\pi_R + \pi)\lambda + (\xi_3 - \beta'\xi_1)\pi\pi_R = 0$$

$$\Rightarrow \lambda = \frac{(\xi_3\pi_R + \pi) \pm \sqrt{(\xi_3\pi_R + \pi)^2 - 4(\xi_3 - \beta'\xi_1)\pi\pi_R}}{2}$$

$$\therefore \lambda_{16} = \frac{(\xi_3\pi_R + \pi) + \sqrt{(\xi_3\pi_R + \pi)^2 - 4(\xi_3 - \beta'\xi_1)\pi\pi_R}}{2}$$

$$\lambda_{17} = \frac{(\xi_3\pi_R + \pi) - \sqrt{(\xi_3\pi_R + \pi)^2 - 4(\xi_3 - \beta'\xi_1)\pi\pi_R}}{2}$$

$$\lambda_{18} = u = 1 - r$$

Our model is locally asymptotically stable if $|\lambda_i| < 1, i = 1, 2, \dots, 18$.

Hence, for $\lambda_1, \lambda_2, \lambda_3$,

$$|(1 + \alpha_1)\pi_S| < 1 \Rightarrow \pi_S = \frac{1}{1 + \alpha_1} \text{ since } \alpha_1 \text{ and } \pi_S \text{ are non-negative}$$

This means that the transition probability for the susceptible class lies between 0 and $\frac{1}{1 + \alpha_1}$. Likewise for

$$\lambda_4, \lambda_5, \lambda_6, 0 < \pi_2 < 1$$

$$\text{For } \lambda_7, \lambda_8 \text{ and } \lambda_9, 0 < \pi_2 < 1$$

$$\text{For } \lambda_{10}, \lambda_{11}, \lambda_{12}, |\pi_3| < 1 \Rightarrow 0 < \pi_3 < 1$$

$$\text{For } \lambda_{13}, \lambda_{14}, \lambda_{15}, |\pi_A| < 1 \Rightarrow 0 < \pi_A < 1 \text{ since } \pi_A \text{ is a transition probability.}$$

This means that the transition probabilities for infection state 1, infection stage 2, infection stage 3 and the AIDS class all lie between 0 and 1.

For λ_{16} ,

$$\left| \frac{(\xi_3\pi_R + \pi) + \sqrt{(\xi_3\pi_R + \pi)^2 - 4(\xi_3 - \beta'\xi_1)\pi\pi_R}}{2} \right| < 1$$

$$\text{Put } x = \xi_3\pi_R + \pi, \quad y = (\xi_3 - \beta'\xi_1)\pi\pi_R$$

$$\text{Hence } \left| x + \sqrt{x^2 - y} \right| < 2$$

$$\Rightarrow -2 - x < \sqrt{x^2 - y} < 2 - x$$

$$\therefore -1 < \frac{\sqrt{x^2 - y}}{x+2} < \frac{2-x}{2+x} < 1 \text{ since } x > 0$$

$$x^2 - y < (2+x)^2 = x^2 + 4x + 4$$

$$-y < 4x + 4 \Rightarrow 4x + y + 4 > 0$$

Substituting for x and y we get

$$4 + 4(4\xi_3\pi_R + \pi) + (\xi_3 - \beta'\xi_1)\pi\pi_R > 0$$

$$\Rightarrow \xi_1(4 + \pi)\pi_R > \pi(\pi_R\beta'\xi_1 - 4) - 4 \Rightarrow \xi_1 > \frac{\pi(\pi_R\beta'\xi_1 - 4) - 4}{(4 + \pi)\pi_R}$$

$$\text{Similarly, for } n_{17}, \xi_1 < \frac{\pi(\pi_R\beta'\xi_1 - 4) - 4}{(4 + \pi)\pi_R}$$

For n_{18}

$$|u| < 1 \Rightarrow |1-r| < 1$$

$$\Rightarrow -1 < 1-r < 1 \Rightarrow -2 < -r < 0$$

$$\text{or } 2 > r > 0 \Rightarrow 0 < r < 2$$

$$\Rightarrow 0 < r < 2$$

This means that the rate of depreciation of stock of capital lies between 0 and 2. Applying the actual definition of π_r , $r = 1, 2, 3$. A. R as in (2.6), these conditions imply,

$$(1) \quad 0 < \pi_1 < \frac{1}{1 + \alpha_1}$$

$$0 < \pi_1 < \frac{1}{1 + \alpha_1} \Rightarrow 0 < \pi_1 = \exp\left(-\int_1^{1+\alpha_1} (\mu + \lambda_1 + \xi_1) ds\right) < \frac{1}{1 + \alpha_1}$$

$$0 < -\int_1^{1+\alpha_1} (\mu + \lambda_1 + \xi_1) ds < \ln(1 + \alpha_1)$$

$$0 > \int_1^{1+\alpha_1} (\mu + \lambda_1 + \xi_1) ds > \ln(1 + \alpha_1)$$

$$\ln(1 + \alpha_1) < \mu + \lambda_1 + \xi_1 \text{ for } i = 1, 2, 3.$$

$$(2) \quad 0 < \pi_1 < 1 \Rightarrow 0 < \exp\left(\int_1^{1+\alpha_1} (\mu + \gamma_{1i}) ds\right) < 1$$

$$\Rightarrow 0 < -\int_1^{1+\alpha_1} (\mu + \gamma_{1i}) ds < 0 \Rightarrow \mu + \gamma_{1i} > 0 \text{ for } i = 1, 2, 3.$$

(3) Similarly, $0 < \pi_2 < 1$ and $0 < \pi_3 < 1$ imply

$$\mu + \gamma_{2i} > 0 \text{ and } \mu + \gamma_{3i} > 0 \text{ for } i = 1, 2, 3$$

$$(4) \quad 0 < \pi_1 < 1 \Rightarrow 0 < \exp\left(-\int_1^{1+\alpha_1} \delta ds\right) < 1 \Rightarrow \delta > 0$$

Summarizing the above analysis, we have the following proposition.

Proposition 2: The conditions for (2.17) to be asymptotically stable are that

- (i) $\ln(1 + \alpha_i) < \mu + \lambda_i + \xi_2 \Rightarrow (1 + \alpha_i) < e^{\mu + \lambda_i + \xi_2} \Rightarrow \alpha_i < e^{\mu + \lambda_i + \xi_2} - 1, i = 1, 2, 3.$
- (ii) $\mu + \gamma_k > 0$ for $k = 1, 2, 3, i = 1, 2, 3$
- (iii) $\delta > 0$
- (iv) $\xi_3 > \frac{1 - \pi}{\pi_R}$
- (v) $0 < r < 2$

3.3 WHEN NOBODY IS INFECTED (DISEASE FREE EQUILIBRIUM)

$$(\lambda_i = 0, I_{1i} = I_{2i} = I_{3i} = A_i = 0)$$

At this point our matrix D will reduce to

$$D' = \begin{pmatrix} (1 + \alpha_1)\pi_S & 0 & 0 & (\alpha + \eta)\pi_S & \alpha\pi & 0 \\ 0 & (1 + \alpha_1)\pi_S & 0 & (\alpha + \eta)\pi_S & \alpha\pi & 0 \\ 0 & 0 & (1 + \alpha_1)\pi & (\alpha + \eta)\pi & \alpha\pi & 0 \\ (\xi_2 + \xi_3)\pi_R & (\xi_2 + \xi_3)\pi_R & (\xi_2 + \xi_3)\pi & -\pi & -\pi & 0 \\ \beta'\pi & \beta'\pi & \beta'\pi & \beta'\pi & -\pi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

3.3.1 The steady state: The steady state is

$$\{S_i^*, 0, 0, 0, 0, R^*, N^*, K^*\} i = 1, 2, 3$$

3.3.2 Stability: We obtain the characteristic equation for the matrix above (D'). Hence

$$|D' - \lambda X| = [(1 + \alpha_1)\pi_S - \lambda]^3 (u - \lambda) [(\xi_3\pi_R - \lambda)(\pi - \lambda) - \beta'\xi_3\pi\pi] = 0$$

$$\Rightarrow x_1 = x_2 = x_3 = (1 + \alpha_1)\pi_S$$

$$x_6 = u$$

$$x^2 - (\xi_3\pi_R + \pi)x + (\xi_3 - \beta'\xi_3)\pi\pi = 0$$

$$x = \frac{(\xi_3\pi_R + \pi) \pm \sqrt{(\xi_3\pi_R + \pi)^2 - 4(\xi_3 - \beta'\xi_3)\pi\pi}}{2}$$

$$\therefore x_4 = \frac{(\xi_3\pi_R + \pi) + \sqrt{(\xi_3\pi_R + \pi)^2 - 4(\xi_3 - \beta'\xi_3)\pi\pi}}{2}$$

$$x_5 = \frac{(\xi_3\pi_R + \pi) - \sqrt{(\xi_3\pi_R + \pi)^2 - 4(\xi_3 - \beta'\xi_3)\pi\pi}}{2}$$

For $x_1, x_2, x_3, (1 + \alpha_1)\pi_S < 1$

$$\Rightarrow 0 < \pi_s < \frac{1}{1 + \alpha_1}$$

$$\text{For } x_6, |u| < 1 \Rightarrow |1 - r| < 1$$

$$\Rightarrow 0 < \frac{r}{2} < 1, \Rightarrow 0 < r < 2$$

This means that

(1) Transition probability from one state to the other in the susceptible class will lie between zero and

$$\frac{1}{1 + \alpha_1} < 1 \Rightarrow 1 + \alpha_1 > 1 \Rightarrow \alpha_1 > 0$$

(2) The depreciation rate will increase (infact, doubled) since $r < 1$ normally.

4. INTERPRETATION OF THE RESULTS AND CONCLUSION

4.1 The main Results of the Study

In this study we divide our population (Nigeria) into eight segments P (population of those between 15 years and 65 years), N (population of those below 15 years), S (the susceptible class), R (the Reservoir or resistant group), I_{1j} , (incubation stage 1), I_{2j} (Incubation stage 2), I_{3j} (Incubation stage 3), A_j (final stage of fully developed AIDS). Each of the S, I_1 , I_2 , I_3 , A components is subdivided into three risk groups, i (heterosexuals, intravenous drug users (IDU) and homosexuals) and each of these risk groups is further divided into three subgroups on rates of acquisition of sexual partners, j (low, medium and high).

Using the above segments a discrete time model of HIV transition and the impact of AIDS on Economic Activities are developed, see (2.17)

It was shown that there exists one and only one non - trivial steady state for the model and the steady state is given by

$$(S_i^0, I_{1i}^0, I_{2i}^0, I_{3i}^0, A_i^0, R^0, N^0, K^0) \quad i = 1, 2, 3$$

The conditions for the stability of the above steady state were found to be as follows:

$$(i) \quad 0 < \pi_1 < \frac{1}{1 + \alpha_1}$$

Which means that the transition probability for the susceptible class lies between zero and $\frac{1}{1 + \alpha_1}$

$$(ii) \quad 0 < \pi_k < 1, \quad k = 1, 2, 3 \text{ A.}$$

Which means that the transition probabilities of the infection stages 1, 2 and the AIDS class lie between 0 and 1.

$$(iii) \quad \xi_3 > \frac{\pi(\pi R \beta_{\xi_1}^- - 4) - 4}{(4 + \pi)\pi R}$$

$$(iv) \quad 0 < r < 2$$

This means that the rate of depreciation of stock of capital lies between 0 and 2. We therefore conclude that the resistant group is very crucial in checking the spread of the epidemics, hence people should heed to the campaigns against HIV/AIDS. Also we conclude that HIV/AIDS has no significant effect on the rate of depreciation of total stock of the nation.

It is also seen from the study that at equilibrium, $S = 0$. That is, the susceptible class will not be wiped out if and only if there are people in the resistant group.

4.2 CONCLUSION

The model developed here is an enhancement of existing models in HIV/AIDS. In fact, this model is the first that combines epidemiological factors with economic factors. The results obtained here may be of interest of experts in the field of Biology and Medicine. The model should help epidemiologists to advice the health authorities to encourage people to heed to the campaigns against HIV/AIDS.

Based on the discussions above, we conclude that:

- (i) Somebody must always be at the risk of contacting HIV/AIDS, ($S = 0$).
- (ii) The depreciation rate, r , of stock of capital is not influenced by the AIDS epidemic at the steady state.
- (iii) $S = 0$ if $R = 0$. This means that the resistant group is crucial. For AIDS not to wipe out the susceptible class R must satisfy $R = 0$.

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