THE SPLINE APPROXIMATION METHOD FOR VOLTERRA'S INTEGRAL EQUATIONS OF THE SECOND KIND

I. O. ISAAC and A. O. UNANAM

ABSTRACT

The use of different numerical methods to evaluate Volterra's integral equations has undoubtedly proved very effective. However, only a little effort or none at all has been made to explore the spline approximations. In this paper, we examine the numerical solutions of Volterra's integral equations of the second kind using cubic spline approximation method. Some numerical results are obtained for the case of difference kernels.

KEYWORDS: Spline approximations, Volterra's integral equations of the second kind.

1. INTRODUCTION

The classical investigations by great mathematicians like Fredholm, Volterra, Hilbert, Tricomi, Muskhelishvili, Mikhlin etc. on the theory of integral equations determined the success of their applications as means of solving problems in diverse areas like the boundary value problems in mathematical physics, the theory of elasticity, aerodynamics, to mention just a few.

As is known, only in exceptional cases are the integral equations amenable to analytic solutions and their uses therefore became possible and even more wide-spread only with the advent of high-speed computer facilities.

In this paper, we are particularly concerned with the development of yet another approximation method based on cubic spline interpolations for solving Volterra's integral equations of the second kind

$$y(x) - \int_a^x K(x,s)y(s)ds = f(x), \quad x \in [a, b) \tag{1.1}$$

with difference or with degenerate kernels. We intend to conclude the investigation by comparing the results of our numerical illustration with the solutions already obtained in Isaac (2006) using mechanical quadrature method for solving the same example.

The justification for the choice of spline in this context is connected with its known cases of producing qualitative approximation results and what is more, the effectiveness in the realization of its algorithms in any computing device (Meinarovish et al, 1974 [a, b]).

Spline is understood to mean a function $S_m(\Delta_m, x)$ which is defined and has continuous (m-1)-st derivatives on the interval [a, b]. On each segment $[x_i, x_{i+1}]$ formed by the network.

$$\Delta_n : a = x_0 < x_1 < \ldots < x_n = b,$$

it coincides with a certain algebraic polynomial of degree not more than m.

Among the known types of splines, an important role is played by the so-called cubic spline, $S_3(x)$. It is noteworthy that its discovery (Stishkin and Subbotin, 1976) marked the beginning of intensive development of the spline concepts, while its influence immediately extended to a wide spectrum of physical and technical problems. One reason may be that any approximation involving the cubic spline can usually be reduced to solving a system of linear equations with a tri-diagonal matrix having a dominant main diagonal (Stishkin and Subbotin, 1976; Marchuk and Agushkov, 1981; Bakhvalov, 1977).

2. Cubic Spline Interpolation

The interpolation by means of cubic spline satisfies the following conditions:

(a) S(x) is continuous and fulfills the conditions of continuity of the derivatives up to order 2 at the points $x_1, x_2, \ldots, x_{N-1}$, i.e. belongs to the class $C^2(a, b)$ of functions;

(b) On each of the subintervals $[x_i; x_{i+1}]$, S(x) is required to be a polynomial of degree 3, that is

I. O. Isaac, Department of Maths/Stats & Computer Science University of Calabar
A. O. Unanam, Department of Maths/Stats & Computer Science, University of Calabar

$$S(x) = S_j(x) = \sum_{i=1}^{3} a_i^{(l)}(x_j - x)^i, \quad j = 1, 2, \cdots, N \tag{2.1}$$

(c) At the knots $\{x_i\}_{j=0}^{N}$, it satisfies the equality

  $S(x_j) = f_j, \quad j = 1, 2, \cdots, N$

(d) The derivative $S''(x)$ satisfies the following boundary conditions:

  $$S''(a) \approx S''(b) = 0 \tag{2.2}$$

  The fulfillment of the above conditions is due to the fact that the spline in equation (2.1) has a concrete form (Olayi, 2000)

$$S(x) = S_j(x) = m_{j-1}\frac{(x_j - x)^3}{6h_j^*} + m_j\frac{(x - x_{j-1})^3}{6h_j^*} + m_{j-1}\left(f_{j-1} - \frac{m_{j-1}(h_j^*)^2}{6}\right)\frac{x_j - x}{h_j^*} +$$
$$+ m_{j-1}\left(f_j - \frac{m_j(h_j^*)^2}{6}\right)\frac{x - x_j - 1}{h_j^*}, \tag{2.3}$$

where

  $h_j^* = x_j - x_{j-1}$,

and

  $m_k = s''(x_k), \quad k = 1, 2, \cdots, N.$

The boundary conditions in (2.2) suggest that

  $m_0 = m_N = 0$

The determination of the values of the remaining quantities $m_1, m_2, \ldots, m_{N-1}$ requires solving a system of linear algebraic equations

  $Am = Hf,$ (2.4)

where the square matrix A of the order $(N-1) \times (N-1)$ has the form

$$A = \begin{pmatrix} \dfrac{h_1^* + h_2^*}{3} & \dfrac{h_2^*}{6} & 0 & \cdots & 0 & 0 \\ \dfrac{h_2^*}{6} & \dfrac{h_2^* + h_3^*}{3} & \dfrac{h_3^*}{6} & \cdots & 0 & 0 \\ 0 & \dfrac{h_3^*}{6} & \dfrac{h_3^* + h_4^*}{3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \dfrac{h_{N-1}^*}{6} & \dfrac{h_{N-1}^* + h_N^*}{3} \end{pmatrix}, \tag{2.5}$$

the rectangular matrix H of the order $(N-1) \times (N-1)$ is defined by

$$H = \begin{pmatrix} \dfrac{1}{h_1^*} & \left(-\dfrac{1}{h_1^*} - \dfrac{1}{h_2^*}\right) & \dfrac{1}{h_2^*} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \dfrac{1}{h_2^*} & \left(-\dfrac{1}{h_2^*} - \dfrac{1}{h_3^*}\right) & \dfrac{1}{h_3^*} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \dfrac{1}{h_{N-1}^*} & \left(-\dfrac{1}{h_{N-1}^*} - \dfrac{1}{h_N^*}\right) & \dfrac{1}{h_N^*} \end{pmatrix}, \tag{2.6}$$

the vector

  $f = (f_0, f_2, \ldots, f_N)^T$

is the free term, while the vector

  $m = (m_1, m_1, \ldots, m_{N-1})^T$

is the unknown.

Matrix A is positively defined and invertible so long as it remains symmetrical and has a dominant main diagonal (Olayi, 2000). This means that the system of the equations in (2.4) is always solvable and produces accurate results with high precisions for the determination of $m_1, m_2, \ldots, m_{N-1}$.

If the interval [a, b] is broken down into uniform knots, that is, the width h_j^* is constant and satisfies the equality

$$h_j^* = h^* = \frac{(b-a)}{N},$$

then the matrices A and H are defined as

$$A = \begin{pmatrix} \frac{2}{3}h^* & \frac{1}{6}h^* & 0 & \ldots & 0 & 0 \\ \frac{1}{6}h^* & \frac{2}{3}h^* & \frac{1}{6}h^* & \ldots & 0 & 0 \\ 0 & \frac{1}{6}h^* & \frac{2}{3}h^* & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \frac{1}{6}h^* & \frac{2}{3}h^* \end{pmatrix}, \tag{2.7}$$

and

$$H = \begin{pmatrix} \frac{1}{h^*} & -\frac{2}{h^*} & \frac{1}{h^*} & \ldots & 0 & 0 & 0 \\ & \frac{1}{h^*} & -\frac{2}{h^*} & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & \ldots & \frac{1}{h^*} & -\frac{2}{h^*} & \frac{1}{h^*} \end{pmatrix}, \tag{2.8}$$

respectively.

3. Application of Cubic Spline

Most known examples involving the use of spline approximations in solving Volterra's integral equations of the second kind are limited to cases where the kernels are given in tabular form. Certainly, to anticipate any meaningful improvement in the solution of these problems, we must be able to device methods of handling functional kernels of arbitrary nature. We are proposing the application of the cubic spline to approximate the values of the kernel of the Volterra's integral equations with difference kernels

$$y(x) - \int_0^x k(x-s)\, y(s)\, ds = f(x), \quad x \in [0;\ b], \tag{3.1}$$

at the knots $\{x_i\}_{j=0}^n$ on the interval [0, b). This gives the approximations

$$K_j = K(x_j), \quad j = 1, 2, \cdots, n.$$

To achieve this, we adapt the approximating equation in (2.3) and rewrite it in the form

$$K(x) = d_{1i}(x_i - x) + d_{2i}(x - x_{i-1})^3 + d_{3i}(x_i - x) + d_{4i}(x - x_{i-1})^3, \tag{3.2}$$

where

$$d_{1i} = \frac{m_{i-1}}{6h_j^*}, \quad d_{2i} = \frac{m_i}{6h_j^*}, \quad d_{3i} = \frac{K_{i-1} - \frac{m_{i-1}(h_j^*)^2}{6}}{h_j^*},$$

$$d_{4i} = \frac{K_i - \frac{m_i(h_j^*)^2}{6}}{h_j^*}, \quad x_{i-k} \leq x \leq x_i, \quad j = 1, 2, \cdots, N. \tag{3.3}$$

Substituting (3.2) into equation (3.1), we obtain a system of equations in respect of $y_i \cong y(x_i)$, $i = 1, 2, \cdots, n$ $(y_0 \cong y(x_0) = f(x_0))$:

$$y(x_i) + \sum_{j=1}^{i} \int_{x_{j-1}}^{x_j} \{d_{1j}[(x_j - x_i) + s]^3 + d_{2j}[(x_i - x_{j-1}) - s]^2 + d_{3j}[(x_j - x_i) + s] +$$

$$+ d_{4j}[(x_i - x_{j-1}) - s]\} \overline{y}(s)ds = f(x_i) \qquad (3.4)$$

xpanding the terms inside the brackets, it becomes clear that the use of spline to approximate difference ernels allows us to transform the original equation into a system of integral equations with degenerate ernels. Applying the property of separable kernels (Myskis, 1979; Hochdtadt, 1973), the system of quations in (3.4) is further transformed into the form:

$$y(x_i) + \sum_{j=1}^{i} \left\{ d_{1j} \left[(x_j - x_i)^3 \int_{x_{j-1}}^{x_j} y(s)ds + 3(x_j - x_i)^2 \int_{x_{j-1}}^{x_j} sy(s)ds + 3(x_j - x_i) \int_{x_{j-1}}^{x_j} s^2 y(s)ds + \right. \right.$$

$$+ \int_{x_{j-1}}^{x_j} s^3 y(s)ds \bigg] + d_{2j} \bigg[(x_i - x_{j-1})^3 \int_{x_{j-1}}^{x_j} y(s)ds - 3(x_i - x_{j-1})^2 \int_{x_{j-1}}^{x_j} sy(s)ds +$$

$$+ 3(x_i - x_{j-1}) \int_{x_{j-1}}^{x_j} s^2 y(s)ds - \int_{x_{j-1}}^{x_j} s^3 y(s)ds \bigg] + d_{3j} \bigg[(x_j - x_i) \int_{x_{j-1}}^{x_j} y(s)ds + \int_{x_{j-1}}^{x_j} sy(s)ds \bigg] +$$

$$+ d_{4j} \bigg[(x_i - x_{j-1}) \int_{x_{j-1}}^{x_j} y(s)ds - \int_{x_{j-1}}^{x_j} y(s)ds - \int_{x_{j-1}}^{x_j} sy(s)ds \bigg\} = f(x_i) \qquad (3.5)$$

eplacing the integrals in (3.5) with the quadrature formulas (Kopchenova and Maron, 1981)

$$\int_{x_{j-1}}^{x_j} y(s)ds \cong (A_{j-1}y_{j-1} + A_j y_j)h,$$

e obtain an expression for the computation of the unknowns

$$y_1 = \frac{A_0 y_0 (d_{11}x_1^3 - d_{21}x_1^3 + d_{31}x_1 - d_{41} - d_{41}x_1)}{A_1 y_1 (d_{11}x_1^3 - d_{21}x_1^3 + d_{31}x_1 - d_{41} - d_{41}x_1)},$$

$$y_i = -\frac{1}{B_i + 1} \{ f_i - h \sum_{j=1}^{i-1} \{ d_{1j} [(x_j - x_i)^3 (A_{j-1}y_{j-1} + A_j y_j) +$$

$$+ 3(x_j - x_i)^2 (x_{j-1}A_{j-1}y_{j-1} + A_j x_j y_j) +$$

$$+ 3(x_j - x_i)(A_{j-1}x_{j-1}^2 y_{j-1} + A_j x_j y_j) + (A_{j-1}x_{j-1}^3 y_{j-1} + A_j x_j^3 y_j)] +$$

$$+ d_{2j} [3(x_i - x_{j-1})^3 (A_j y_{j-1} + A_j y_j) -$$

$$- 3(x_i - x_{j-1})^2 (A_{j-1}x_{j-1}y_{j-1} + A_j x_j y_j) + 3(x_i - x_{j-1})(A_{j-1}x_{j-1}^2 y_{j-1} + A_j x_j^2 y_j) +$$

$$(A_{j-1}x_{j-1}y_{j-1} + A_j y_j) - (A_{j-1}x_{j-1}y_{j-1} + A_j x_j y_j)] \} \} .$$

here

$$B_i = d_{2i}(h^3 A_i - 3h^2 x_i + 3hx_i^2 - 3x_i^3)h + d_{4i}(hA_i - A_i - x_i y_i h), \quad i = 2, 3, \cdots, n.$$

Illustration

To demonstrate the effectiveness of the method, we experiment the algorithm for the determination f the numerical solutions of the equation

$$y(x) = e^{x^2 + 2x} + 2\int_{a}^{x} e^{(x^2 - s^2)} y^2(s)ds, \quad x \in [0.00; 0.08], \qquad (4.1)$$

here the exact solution is given as

$$y(x) = e^{x^2 + 2x}(1 + x).$$

e partition the interval $[0.00; 0.08]$ into four subintervals by the equidistant knots $x_j = jh$, where = 1, 2, 3, 4 and the step-length $h = 0.02$.

In respect of the zero approximation, we have the function $y_0(x) = e^{x^2}$. Hence, we determine the values $y_{0j} = y_0(x_j)$ and $F_{0j} = F[y_0(x_j)]$, where $F[y(x)] = y^2(x)$ as follows:

$y_{00} = 1$; $y_{01} = 1.000401$; $y_{02} = 1.001601$; $y_{03} = 1.003606$; $y_{04} = 1.006420$;

$F_{00} = 1$; $F_{01} = 1.000802$; $F_{02} = 1.003205$; $F_{03} = 1.007225$; $F_{04} = 1.012881$.

By virtue of the set of the knots $x_j = 0.00$; 0.02; 0.04; 0.06; 0.08 and the corresponding values F_{0j}, $j = 1, 2, 3, 4$, we immediately construct the relevant cubic spline as follows, bearing in mind of course, equation (2.3)

$$G_0(x) = S[F(y_0)] = m_{j-1} \frac{(x_j - x)^3}{6h} + m_j \frac{(x - x_{j-1})^3}{6h} + \left(F_{0,j-1} - \frac{m_{j-1} h^2}{6} \right) \frac{x_j - x}{h} +$$

$$\left(F_{0j} - \frac{m_{j-1} h^2}{6} \right) \frac{x - x_{j-1}}{h}, \ x_{j-1} \leq x \leq x_j, \ j = 1, 2, 3, 4. \tag{4.2}$$

Further, we determine the values m_1, m_2, m_3 ($m_0 = m_4 = 0$) from statement (2.4), which, in this case, translates to the system

$$\begin{pmatrix} \frac{2h}{3} & \frac{h}{6} & 0 \\ \frac{h}{6} & \frac{2h}{3} & \frac{h}{6} \\ 0 & \frac{h}{6} & \frac{2h}{3} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{h} & -\frac{2}{h} & \frac{1}{h} & 0 & 0 \\ 0 & \frac{1}{h} & -\frac{2}{h} & \frac{1}{h} & 0 \\ 0 & 0 & \frac{1}{h} & -\frac{2}{h} & \frac{1}{h} \end{pmatrix} \begin{pmatrix} F_{00} \\ F_{01} \\ F_{02} \\ F_{03} \\ F_{04} \end{pmatrix}.$$

or

$$\begin{cases} \frac{2h}{3} m_1 + \frac{h}{6} m_2 = \frac{1}{h} F_{00} - \frac{2}{h} F_{01} + \frac{1}{h} F_{02}, \\ \frac{h}{6} m_1 + \frac{2h}{3} m_2 + \frac{h}{6} m_3 = \frac{1}{h} F_{01} - \frac{2}{h} F_{02} + \frac{1}{h} F_{03}, \\ \frac{h}{6} m_2 + \frac{2h}{3} m_3 = \frac{1}{h} F_{02} - \frac{2}{h} F_{03} + \frac{1}{h} F_{04}. \end{cases}$$

For the solutions of the system, we obtain the values
$m_1 = 5.13830$, $m_2 = 3.46179$ $m_3 = 5.26955$.

thus, providing the possibility for a concrete representation of equation (4.2). In particular, on the interval $[x_0; x_1]$ the cubic spline has the form

$$G_0(x) = \frac{1}{6h} \left[6F_{00} x_1 + \left(6F_{01} - 6F_{00} - m_1 h^2 \right) x + m_1 x^3 \right] \ x \in [0.00; 0.02]$$

This enables us to determine the value of the first approximation at the point $x_1 = 0.02$ as follows:

$$y_1(x_1) = y_0(x_1) + 2 \int_a^{x_1} e^{(x^2 - s^2)} G(s) ds$$

$$= e^{0.02^2 - 2(0.02)} + 2e^{0.02^2} \int_0^{0.02} e^{-s^2} \frac{1}{6h} \left[6F_{00} x_1 + \left(6F_{01} - 6F_{00} - m_1 h^2 \right) s + m_1 s^3 \right] ds$$

$$= 1.08125.$$

On the interval $[x_1; x_2]$, equation (4.2) has the form

$$G_1(x) = \frac{1}{6h} \left[m_1 h^3 + 6F_{01} x_2 - 6F_{02} x_1 + \left(3m_2 x_1^2 - 3m_1 x_2^2 - 6F_{01} + 6F_{02} + m_1 h^2 - m_2 h^2 \right) x +$$

$$+ \left(3m_1 x_2 - 3m_2 x_1 \right) x^2 + \left(m_2 - m_1 \right) x^3 \right]. \ x \in [0.02; 0.04]$$

Thus, we calculate the value of the first approximation at the point $x_2 = 0.04$ as follows:

$$y_1(x_2) = y_0(x_2) + 2\int_a^{x_1} e^{(x^2-s^2)}G(s)ds = 2e^{x_2^2}\left(\frac{e^{2x_2}}{2} + \int_0^{x_1} e^{-s^2}G_0(s)ds\right)$$

$$= 2e^{x_2^2}\left(\frac{e^{2x_2}}{2} + \int_0^{x_1} e^{-s^2}G_0(s)ds + \int_{x_1}^{x_2} e^{-s^2}G_0(s)ds\right).$$

Applying the corresponding limits of integration on the integrals under the bracket, we obtain

$$y_1(x_2) = 2e^{0.04^2}(0.520405 + 0.0199712 + 0.01995858) = 1.12112.$$

On the interval $[x_1; x_2]$, equation (4.2) becomes

$$G_0(x) = \frac{1}{6h}\big[24m_2h^3 - 6m_3h^3 + 6F_{02}x_3 - 6F_{03}x_2 + (3m_3x_2^2 - 3m_2x_3^2 - 6F_{02} + 6F_{03} +$$

$$+ m_2h^2 - m_3h^2)x + (3m_2x_3 - 3m_3x_2)x^2 + (m_3 - m_2)x^3\big], \quad x \in [0.04; \ 0.06].$$

Hence, we obtain the first approximation at the point $x_3 = 0.06$ as follows:

$$y_1(x_3) = y_0(x_3) + 2\int_a^{x_3} e^{(x^2-s^2)}G(s)ds = 2e^{x_3^2}\left(\frac{e^{2x_3}}{2} + \int_0^{x_3} e^{-s^2}G_0(s)ds\right)$$

$$= 2e^{x_3^2}\left(\frac{e^{2x_3}}{2} + \int_0^{x_1} e^{-s^2}G_0(s)ds + \int_{x_1}^{x_2} e^{-s^2}G_0(s)ds + \int_{x_2}^{x_3} e^{-s^2}G_0(s)ds\right).$$

Substituting for x_1, x_2, and x_3 in the limits of integration, we obtain

$$y_1(x_3) = 1.25710.$$

Finally, on the interval $[x_3; x_4]$, the cubic spline equation (4.2) takes the form

$$G_0(x) = \frac{1}{6h}\big[60m_3h^3 + 6F_{03}x_4 - 6F_{04}x_3 + (6F_{04} - 3m_3x_4^2 - 6F_{03} + m_3h^2)x +$$

$$+ 3m_3x_4x^2 - m_3x^3\big], \quad x \in [0.06; \ 0.08].$$

The first approximation at the point $x_4 = 0.08$ is therefore determined by the formula

$$y_1(x_4) = y_0(x_4) + 2\int_0^{x_4} e^{(x^2-s^2)}G(s)ds = 2e^{x_4^2}\left(\frac{e^{2x_4}}{2} + \int_0^{x_4} e^{-s^2}G_0(s)ds\right)$$

$$= 2e^{x_4^2}\left(\frac{e^{2x_4}}{2} + \int_0^{x_1} e^{-s^2}G_0(s)ds + \int_{x_1}^{x_2} e^{-s^2}G_0(s)ds + \int_{x_2}^{x_3} e^{-s^2}G_0(s)ds + \int_{x_3}^{x_4} e^{-s^2}G_0(s)ds\right).$$

Thus,

$$y_1(x_4) = 1.34040.$$

We summarize our illustration by placing the above solutions side by side with those obtained in Isaac (2006) by mechanical quadrature method. Better still, we arrange them tabular form as follows:

Table 4.1

$y(x)$	Exact solution	Mechanical quadrature method	Cubic spline approximation method
$y(0.00)$	1.00000	1.00000	1.00000
$y(0.02)$	1.06205	1.10570	1.08125
$y(0.04)$	1.12842	1.20295	1.12112
$y(0.06)$	1.19946	1.31308	1.25710
$y(0.08)$	1.27553	1.4387	1.34040

5.   CONCLUSION

Clearly, the results as presented in the above table reveals that the cubic spline approximation method leads to more improved solutions than the mechanical quadrature method. What is more, it is also

discovered that it is much easier to implement the spline approximation method in computing devices than the mechanical quadrature method.

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