GLOBAL JOURNAL OF MATHEMATICAL SCIENNCES VOL. 7, NO. 1, 2008: 63 – 68

NUMERICAL ALGORITHM FOR DIGITAL IMAGE ENHANCEMENT AND NOISE MINIMIZATION.

L. N. EZEAKO and K. R. ADEBOYE

(Received 1 August 2007; Revision Accepted 17 March 2008)

ABSTRACT

We adopt the approach of Vogel and Oman, 1998 and introduce a Lagrange multiplier Ibiejugba, 1985, to obtain an appropriate discrete energy which we minimize, in order to minimize equivalently, the unwanted vibration (noise) associated with a digitally transmitted image. An iterative algorithm is developed for this minimization and the convergence of the algorithm is proved analytically.

KEY WORDS: Digital Image Enhancement, Noise Minimisation

1. INTRODUCTION

An image is a bounded gray level function, $g:\Omega \to [0,1]$, where Ω is a "screen" which is usually an open domain in R^2 e.g.a rectangle (0,1) x (0,1) $g(x) = Au(x) + n(x)$, in practice, where A is a linear operator say, from $L^2(\Omega)$ to $L^2(\Omega)$, u(x) is a good image and n(x) is a vibration (noise) Rudin, Osher and Fatemi, 1992.

We would need to find the best function u among all possible u, satisfying:

$$(1)\quad \begin{cases} \int_\Omega Au(x) - g(x)dx = 0 \\ \int_\Omega \left| Au(x) - g(x)^2 \right| = \sigma^2 \end{cases}$$

Where 0 is the mean and σ^2 is the variance. We adopt the approach of Rudin, Osher and Fatemi, 1992, who proposed the "total variation" of the function of u as a measure of the optimality of the image. This criterion is approximately the integral $\int_\Omega |\nabla u(x)dx|$

The main advantage is that this integral can be defined for functions which have discontinuities along hypersurfaces (in two-dimensional images, along one-dimensional curves). This is essential to get a correct representation of the edges in an image to facilitate pattern recognition etc.

The main task is to minimize the integral $\left\{ \int_\Omega |\nabla u(x)| dx : u\, satisfies (1) \right\}$ ---------------- (P1)

2. A DISCRETE ENERGY APPROACH TO THE MAIN TASK

We consider problem (P_1) in dimension 2 and endeavour to compute a solution. We adopt the approach of Vogel and Oman, 1996, 1998. We assume the existence of a Lagrange multiplier $\lambda > 0$ (see Ibiejugba, 1985) such that (P_1) is equivalent to the problem:

$$Min\left\{ |Du|(\Omega + \lambda \int_\Omega |Au(x) - g(x)|^2 dx \right\}; u \in BV(\Omega) \text{ --- } \quad \text{---} \quad (P_2)$$

2.1 Assumptions

(i) The operator A, satisfies AI = I (i.e. the image of a constant function is the same function)

(ii) The initial data satisfies $\int_\Omega |g(x) - f_\Omega g|^2 dx \geq \sigma^2$

L. N. Ezeako, Department of Mathematics/ Computer Sciences, Federal University of Technology, Minna Nigeria
K. R. Adeboye, Department of Mathematics / Computer Science, Federal University of Technology Minna Nigeria

(iii) There exists a $\bar{u}$ satisfying equation (1) such that $|Du|(\Omega) < \infty$

2.2 Discretization

By making all the assumptions in section 2.1 the minimizer of (P_2) automatically satisfies $\int_\Omega Au = \int_\Omega g$ see Chambolle and Lions, 1997, for details. We discretise (P_2) assuming that u and g are discretised on the same square Lattice, i,j = 1,...............,L. The functions u and g are thus approximated by the discrete matrices.

$$U = (U_{ij})1 \le i, j \le L \; and \; G = (G_{ij})1 \le i, j \le L$$

The term $\lambda \int_\Omega |Au(x) - g(x)|^2 dx$ is replaced by a term $\lambda \sum_{i,j} |(AU)_{i,j} - g_{i,j}|^2$ in this discrete setting. Hence A denotes a linear operator of $R^N \to R^{LxL}$ and $(AU)_{i,j}$ is the component i,j of AU. The discrete energy we thus need to minimize is

$$E(U) = \sum_{i,j} (|U_{i+1,j} - U_{i,j}| + |U_{i,j+1} - U_{i,j}| + \lambda \sum_{i,j} |(AU)_{i,j} - g_{i,j}| \qquad (P_3)$$

2.3 Remark

Our first reaction is to minimize (P_3) by the gradient method e.g. CGM and ECGM (see Ibiejugba, 1985, Ibiejugba and Abiola 1985 a & b). But the strong nonlinearity of (P_3) and moreso the derivative $D_U E$, pose serious problems. The simplest of these problems is the nonexistence of the derivative of the absolute value $|x|$ at x = 0 [Even though we can overcome this problem by replacing $|x|$ with $\sqrt{\beta + x^2}$, where β is a small parameter, the overall minimization process is cumbersome].

2.4 The Minimization Method

We adopt a method that is common in the image processing literature, (see Chambolle, 1997, Rudin et al., 1992, for example). Observe that for every $x \in \Re, x \neq 0, |x| = \min_{v>0}(\frac{v}{2}x^2 + \frac{1}{2v})$, the minimum being reached for $v = \frac{1}{|x|}$ we thus introduce the function f(x,v) = $\frac{vx^2}{2} + \frac{1}{2v}$ and a new field

$V = (V_{i+\frac{1}{2},j})_{1\le i\le L, 1\le j\le L} \cup ....(V_{i,j+\frac{1}{2}})_{1\le i\le L, 1\le j\le L} \in \Re^{L(L-1)+L(L-1)}$ (of positive real numbers)

and a new energy.

$$F(U,V) = \sum_{i,j} (f(|U_{i+1,j} - U_{i,j}|, V_{i+\frac{1}{2},j}) + ...f(|U_{i,j+1} - U_{i,j}|, v_{i,j+\frac{1}{2}})) + \lambda \sum_{i,j} |(AU)_{i,j} - g_{i,j}|^2$$

$$= \sum_{ij} (\frac{1}{2} V_{i+\frac{1}{2},j} |U_{i+1,j} - U_{i,j}|^2 + \frac{1}{2} V_{i,j+\frac{1}{2}} |U_{i,j+1} - U_{i,j}|^2 + ...\frac{1}{2V_{i+\frac{1}{2},j}} + \frac{1}{2V_{i,j+\frac{1}{2}}} + \lambda \sum_{i,j} |(AU)_{i,j} - g_{i,j}|^2$$

and we notice that, $\min_V F(U,V) = E(U)$, the minimum being reached for

$$V_{i+\frac{1}{2},j} = \frac{1}{|U_{i+1,j} - U_{i,j}|} (or \; at + \infty \; if \; U_{i+1,j} = U_{i,j}) \text{ and}$$

$$V_{i,j+\frac{1}{2}} = \frac{1}{|U_{i,j+1} - U_{i,j}|}$$

We choose some starting values U^0, V^0 and compute for every n > 1

$$U^n = \arg \min_U F(U, V^{n-1})$$

And

$$V^n = \arg \min_V F(U^n, V)$$

The idea is that as n becomes large, U^n will converge to the minimizer of the problem (P_3). This is actually true if we slightly modify this algorithm (and the function E(U) which we minimize)

So we choose $\varepsilon > 0$ and introduce the convex closed set

$$K_\varepsilon = \left\{ V : \varepsilon \le V_{i+\frac{1}{2},j} \ and \ldots \varepsilon \le V_{i,j+\frac{1}{2}} \le \frac{1}{\varepsilon}, \forall i,j \right\} in\, R^M$$

(M = (L-1) x L + L x(L-1))

We define a new energy $E_\varepsilon(U) = \min_{V \in K_\varepsilon} F(U,V)$

It is ea sy to compute E_ε explicitly because;

$$E_\varepsilon = \sum_{i,j} (j_\varepsilon(U_{i+1,j} - U_{i,j}) + j_\varepsilon(U_{i,j+1} - U_{i,j}) + \lambda \sum_{i,j} |(AU)_{i,j} - g_{i,j}|^2$$

$$\text{where } j_\varepsilon(x) = \min_{\varepsilon \le v \le \frac{1}{\varepsilon}} f(x,v) = \begin{cases} \frac{1}{2\varepsilon}x^2 + \frac{\varepsilon}{2} & if\, |x| \le \varepsilon \\ |x| & if\, \varepsilon \le |x| \le \frac{1}{\varepsilon} \\ \frac{\varepsilon}{2}x^2 + \frac{1}{2\varepsilon} & if\, |x| \ge \frac{1}{\varepsilon} \end{cases}$$

Define

$$\phi_\varepsilon(x) = \left(\varepsilon \vee \frac{1}{|x|}\right) \wedge \frac{1}{\varepsilon} = \left(\frac{1}{|x|} if\, \varepsilon \le |x| \le \frac{1}{\varepsilon}, \frac{1}{\varepsilon} if\, |x| \le \varepsilon, and\, \varepsilon\, if\, |x| \ge \frac{1}{\varepsilon}\right)$$

Then $\phi_\varepsilon(x)$ is the unique value in $\left[\varepsilon, \frac{1}{\varepsilon}\right]$ such that $j_\varepsilon(x) = f(x, \phi_\varepsilon(x)$

We deduce that the unique $V \in K_\varepsilon$ for which

$E_\varepsilon(U) = \min_{k_\varepsilon} F(U,.) = F(U,V) is\, given by\, V_{i+\frac{1}{2},j} = \phi_\varepsilon(x_{i+1,j)} - x_{i,j})$ and

$V_{i,j+\frac{1}{2}} = \phi_\varepsilon(x_{i,j+1} - x_{i,j})$ for every i,j

In this case, we set $\phi_\varepsilon(U) = V$. This defines a continuous function $\phi_\varepsilon : R^N \to K_\varepsilon \subset R^M$

The Algorithm, now consists in computing for every n ≥ 1, the starting values U^0, V^0 being chosen;

$$U^n = \arg.\min_U F(U, V^{n-1})$$

and

$$V^n = \arg.\min_{V \in k_\varepsilon} F(U^n, V) = \phi_\varepsilon(U^n)$$

3. Analytical Proof of the Convergence of the Numerical Algorithm for Noise Minimization

i.e. $U^n = \arg.\min_U F(U, V^{n-1})$ And

$$V^n = \arg.\min_{V \in K_\varepsilon} F(U^n, V). = \Phi_\varepsilon(U^n)$$

Proof

Let I_N be the vector in R^N defined by $(I_N)ij = 1$ for every 1≤i, j≤L (where N = L x L is the dimension of the space e.g. the metric space, where U resided).

We assume that the image of a constant function is the same function That is, given the linear operator A, $AI_N = I_N$.

Conjecture

There exist $\bar{U}, \bar{V} = \Phi_\varepsilon(\bar{U})$

such that as n→ ∞, $U^n \to \bar{U}\ and\, V^n \to \bar{V}\ and\, \bar{U}\ is (the) \min imizer\, of\, E_\varepsilon$

Proof of Conjecture

Lemma 1: We claim that there exists $0 < \alpha < \beta$ such that the second derivatives $D^2_{U} F\ and\ D^2_{V} F$ satisfy

$$\alpha I_N \le D^2_{UU}F(U,V) \le \beta I_N \ and\ \alpha I_M \le D^2_{VV}F(U,V) \le \beta I_M$$

for every

$U \in K_\varepsilon$ that is $U \in R^N, V\varepsilon K_\varepsilon, \xi \in R^N$ and $\eta \in R^M$, we have ,

$$\alpha|\xi|^2 \le \langle D^2_{UU}F(U,V)_{\xi\xi}\rangle \le \beta|\xi|^2$$

and

$$\alpha|\eta|^2 \le \langle D^2_{VV}F(U,V)_{\eta\eta}\rangle \le \beta I_M|\eta|^2$$

Proof (see Vogel and Oman, 1998.) We also recall the following "Poincare inequality" (in finite dimension): there exist a constant C > 0 such that for every $\xi \in R^N = R^{l \times l}$ such that $\sum_{i,j}\xi_{i,j} = 0$

$$\text{(3.1)} \qquad \sum_{l\le i,j\le l}|\xi_{i,j}|^2 \le C\left(\sum_{l\le i<l,j}|\xi_{i+1,j} - \xi_{i,j}|^2 + \sum_{i,1\le j<l}|\xi_{i,j+1} - \xi_{i,j}|^2\right)$$

We note that for every $U,V \in K_\varepsilon$ and $\xi \in R^N$,

$$\langle D^2_{UU}F(U,V)_{\xi\xi}\rangle = \sum_{i,j}\left(V_{i+\frac{1}{2},j}|\xi_{i+1,j} - \xi_{i,j}|^2 + V_{i,j+\frac{1}{2}}|\xi_{i,j+1} - \xi_{i,j}|^2\right) + |A\xi|^2$$

$$\ge \varepsilon\sum_{i,j}\left(|\xi_{i+1,j} - \xi_{i,j}|^2 + |\xi_{i,j+1} - \xi_{i,j}|^2\right) + |A\xi|^2$$

In particular, letting $m(\xi) = (\frac{1}{N})\sum_{i,j}\xi_{i,j}$ be the average of ξ we have (since $AI_N = I_N$)

$$\langle D_{UU}F(U,V)_{\xi,\xi}\rangle \ge |A\xi| = |A(\xi - m(\xi)I_N) + m(\xi)I_N| \ge |m(\xi)I_N| - |A||\xi - m(\xi)I_N|.$$

But by equation (3.1)

$$|\xi - m(\xi)I_N|^2 \le c\sum_{i,j}\left(|\xi_{i+1,j} - \xi_{i,j}|^2 + |\xi_{i,j+1} - \xi i,j|^2\right) \le \left(\frac{1}{\varepsilon}\right)\langle D^2_{UU}F(U,V)_{\xi,\xi}\rangle.$$

Therefore $|m(\xi)I_N| \le c\sqrt{\langle D^2_{UU}F(U,V)_{\xi,\xi}\rangle}$ (here c denotes any positive constant that does not depend on U,V,ξ). Moreover, by using equation (3.1) again,

$$c\langle D^2_{UU}F(U,V)_{\xi,\xi}\rangle \ge |\xi - m(\xi)I_N|^2.$$

Since I_N and $\xi - m(\xi)I_N$ are orthogonal we deduce that $|\xi|^2 \le c\langle D^2_{UU}F(U,V)_{\xi,\xi}\rangle$.

Lemma 2: For every n ≥ l

$$E_\varepsilon(U^{n-1}) - E_\varepsilon(U^n) \ge \frac{\alpha}{2}(|U^{n-1} - U^n|^2 + |V^{n-1} - V^n|^2)$$

Proof: For every n ≥ l

$D_U F(U^n, V^{n-1}) = 0$ while

$\langle D_V F(U^n,V^n), V - V^n\rangle \ge 0$ for every $V \in K_\varepsilon$

By Lemma1, we deduce that;

$$F(U^n,V^{n-1}) = F(U^n,V^n) + \langle D_V F(U^n,V^n), V^{n-1} - V^n\rangle$$

$$+ \int_0^1(1-t)\langle D^2VVF(U^n,V^n + t(V^{n-1} - V^n))(V^{n-1} - V^n), V^{n-1} - V^n\rangle dt$$

$$\ge F(U^n,V^n) + \frac{\alpha}{2}|V^{n-1} - V^n|^2.$$

In a similar way, we prove that

$$F(U^{n-1},V^{n-1}) \ge F(U^n,V^{n-1}) + \frac{\alpha}{2}|U^{n-1} - U^n|^2.$$

Since $E_\varepsilon(U^n) = F(U^n,V^n)$, this lemma is proved.

Remark:

By construction, the sequence

$E_\varepsilon(U^n) = F(U^n, V^n)$ must decrease and it is bounded from below. It goes to some constant e and $E_\varepsilon(U^{n-1}) - E\varepsilon(U^n) \to 0$

Thus $U^{n-1} - U^n$ $and V^{n-1} - V^n$ go to zero as $n \to \infty$.

Also from Lemma 1, we notice that

E_ε is coercive, which implies that for every c>0, the set { $E_\varepsilon \le c$} is bounded in R^N. It is also closed and hence, compact. Thus we may extract a subsequence U^{nk} and find a $\bar{U} \in R^N$ such that as $k \to \infty, U^{nk} \to \bar{U}$

By continuity $V^{nk} = \Phi_\varepsilon(U^{nk}) \to \Phi_\varepsilon(\bar{U}), and\ we\ let\ \bar{V} = \Phi_\varepsilon(\bar{U})$.

We also have $D_U F(U^{nk}, V^{nk-1})=0$ and since $V^{nk-1} - V^{nk} \to 0$ (by lemma2)

$V^{nk-1} \to \bar{V}$, so that by continuity, $D_t F(\bar{U}, \bar{V}) = 0$

Proof of Conjecture: Let $h \in R^N$ and t>0

Letting $V_t = \Phi_\varepsilon(\bar{U} + th) \to \bar{V}\ as\ t \to 0$

We have $E_\varepsilon(\bar{U} + th) - E_\varepsilon(\bar{U}) = F(\bar{U} + th, \Phi_\varepsilon(\bar{U} + th)) - F(\bar{U} + \bar{V})$

$= (F(\bar{U} + th, V_t) - F(\bar{U}, V_t)) + (F(\bar{U}, V_t) - F(\bar{U}, \bar{V}))$

Since $V_t \in K\varepsilon$,

$$F(\bar{U}, V_t) \ge F(\bar{U}, \bar{V}), so\ that\ E_\varepsilon(\bar{U} + th) - E_\varepsilon(\bar{U}) \ge F(\bar{U} + th, V_t) - F(\bar{U}, V)$$

Hence, $F(\bar{U} + th, V_t) - F(\bar{U}, V_t) = t\langle D_t F(\bar{U}, V_t), h\rangle + \int_0^1 (t-s)\langle D_{t,t}^2 F(\bar{U}, V_t)h, h\rangle dt$

and $\int_0^1 (t-s)\langle D_{t,t}^2 F(\bar{U}, V_t)h, h\rangle dt \le \beta t^2 |h|^2 / 2$,

$= \lim_{t \to 0} \frac{E_\varepsilon(\bar{U} + th) - E_\varepsilon(\bar{U})}{} \ge \langle D_t F(\bar{U}, \bar{V}), h\rangle = 0$

Since h is arbitrary, $D_t E_\varepsilon(\bar{U}) = 0$

4. CONCLUSION

Since E_ε *is* strictly convex → for every U, U' and

$0 < \theta < 1, E_\varepsilon(\theta U + (1-\theta)U') < \theta E_\varepsilon(U) + (1-\theta)E_\varepsilon(U')$

Unless $U = U'$ it has a unique minimizer characterized by the equation $D_U E = 0$. We deduce that $\bar{U}$ is the UNIQUE MINIMIZER OF E_ε. This achieves the proof of our CONJECTURE

By the uniqueness of this minimizer, any subsequence of (U^n) must converge to the same value $\bar{U}$, so that the whole sequence U^N converges to $\bar{U}$.

Similarly, V^n converges to $\bar{V}$.

REFERENCES

Chambolle, A and Lions, P.L., 1997 Image Recovery Via Total Variation Minimization And Related Problems. Numer Math., 76(2)167-188

Ibiejugba, M. A., 1985 The Ingenuity Of The Method Of Multipliers in Solving Optimization Problems Advances In Modeling and Simulation Reviews, Vol 1, (4): 11-22

Ibiejugba, M. A. and Abiola, B., 1985a. On The Convergence Rate Of The Congruence Gradient Method Advances In Modeling and Simulation Vol 2 No 1 pp 47-56

Ibiejugba, M. A and Abiola, B., 1985b Minimization By Congruents Advances In Modeling and Simulation. Vol 4, (2): 33-44.

Rudin, L., Osher, S. J. and Fatemi, E., 1992. Nonlinear Total Variation Based Noise Removal Algorithms. Physica D., 60: 259-268.

Vogel, C. R. and Oman, M. E., 1996. Iterative Method for Total Variation Denoising. SIAM J. Sc. Comput, 17(1), 227-238.

Vogel, C. R. and Oman, M. E., 1998. Fast, Robust Total Variation-based Reconstruction of Noisy, Blurred Images IEEE Trans. Image Process, 7(6): 813-824.