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ON TOTAL VARIATION METHOD FOR DIGITAL IMAGE ENHANCEMENT, DENOISING AND DEBLURRING

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ABSTRACT

In this paper, we define the necessity for recourse to Total Variation method in digital image filtering and we establish the existence of a refined image in the space of Bounded Variation Functions BV(Ω), for a given "screen" Ω

KEY WORDS: Image (signal), Vibration (Noise), Total Variation, Functions with Bounded Variation

1. PRELIMINARIES

Noise is present in virtually all signals. In some situations it is negligible, in other situations, it all but obliterates the signal of interest. Removing unwanted noise from signals has historically been a driving force behind the development of signal processing technology Frieden, 1975, Mersereau et al., 1975. It continues to be a major application for both analogue and digital processing systems.

Noise/Vibration are random background events which have to be dealt with in every system processing real signals. The absence or negligible presence of noise in any image processing system, facilitates pattern recognition Frieden, 1975.

Noise/Vibration are not part of the ideal signal. They may be caused by a wide range of sources, e.g variation in the detector, sensitive environmental variations, the discrete nature of radiation, transmission or quantization errors, e.t.c. Lee, 1980.

1.1 THE DISCRETE SETTING

An image (signal) G, is a matrix G = (g_{ij}) 1 ≤ i, j ≤ n of gray level value in [0,1] G ≡ the sum of a "perfect world" unknown image u = (u_{ij}) 1 ≤ i, j ≤ n + an additive Gaussian noise N = (N_{ij}) 1 ≤ i, j ≤ n, where all the N_{ij} are independent and have mean 0 and variance σ^2

i.e. G = u + N (See Rudin, Osher and Fatemi, 1992)

1.2 The Continuous Setting

An image is a bounded gray level function g: Ω → [0,1] where Ω is the "screen" which is usually an open domain in R^2, e.g a rectangle (0,1) x (0,1).

$g(x) = u_{(x)} + n_{(x)}$

where $u_{(x)}$ ≡ a good image

$n_{(x)}$ = an oscillation which we will like to remove.(See Rudin et al 1992)

We assume that:

$$\int_\Omega n(x)dx = 0 - the\ mean$$

$$\int_\Omega n(x)^2 dx = \sigma^2 = the\ variance;$$

(is known or is computable)

This initial model is actually simplistic. An image usually has all sorts of corruptions in practice. The correct model is

$g_{(x)} = Au_{(x)} + n_{(x)}$ where A is a linear operator say, from $L^2(\Omega)$ to $L^2(\Omega)$

$$Au_{(x)} = e * u_{(x)} = \int e(x-y)u_{(x)}dy \equiv a\ blur\ or\ convolution$$

e is a non – negative constant)

1.3 Problem

Suppose that one knows the function g and an estimation of A and σ^2, is it possible to obtain a good estimation of u? We show how to obtain the best estimation of 'u' in the following discussion

2. DISCUSSION

The first idea would be to compute $A^{-1}g = u + A^{-1}n$. However, this is not feasible in practice. The operator A is often not invertible, or its inverse is impossible to compute. For example, consider the case where Au = e*u.

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Remark

The first assertion is a consequence of theorem 4 2 The second assertion means that if a sequence of function $(u_j)_{j\geq 1}$ bounded in BV(Ω), i e $\sup_j \|u_j\|_{L^1(\Omega)} + |Du_j|(\Omega) < \infty$ then we can extract a subsequence u_{jk} and there exists a function $u \in$ BV(Ω) such that, as

$k \to \infty$, $Du_{jk} \to Du$ weakly --* as a measure and $u_{jk} \to u$ strongly in $L^p(\Omega)$ for every $p < \frac{N}{N-1}$

Theorem 4.4 (Approximation By Smooth Functions)

Let $u \in$ BV (Ω), then there exist a sequence $(u_n)_{n\geq 1}$ $C^\infty(\Omega)$ such that as $n \to \infty$, $u_n \to u$ in $L^1(\Omega)$, $Du_n \to Du$ weakly $*$ as measures, $|Du_n| = \int_\Omega |\nabla u_n(x)|dx \to |Du|(\Omega)$.

5. Existence For The Total Variation Approach

We now use the result of theorems 4.1 and 4 2 to establish the existence of problem (P_1)

The existence of (P_1) is ensured in dimensions N = 1 or N = 2, provided the following conditions are satisfied

1 The operator A satisfies A1 = I (i.e the image of a constant function is the same function).

2. The initial data satisfies $\int_\Omega |g(x) - f_\Omega g|^2 dx > \sigma^2$

3. There exist a $\bar{u}$ satisfying equation (2.1) such that $|D\bar{u}|(\Omega) < \infty$

Remarks:

The first assumption is not absolutely necessary (we only need that A1 ≠ 0), but simplifies the proof a lot. It is obviously satisfied if A corresponds to a convolution with a kernel of integral 1 (Au = e*u, $\int e = 1$) (provided the boundary effects are correctly treated)

The second assumption is needed. We observe that if the model g = Au + n is correct, then it should be satisfied (when n is rapidly oscillating so that $\int_\Omega Au + n \approx 0$.

The third assumption implies that $I = \inf\{|Du|(\Omega)$: u satisfies (2.1).}<∞, otherwise any u in equation (2.1) is a solution But then, the problem would be of little interest. In the general continuous setting, the existence of such a $\bar{u}$ is not absolutely obvious

5.1 Proof of Existence

See Chambolle and Lions, 1997 We consider a minimizing sequence (Un) n ≥ 1 for problem (P_1), of Functions U_n that all satisfies equation (2.1) and such that $|Du_n|(\Omega) \to I$ as $n \to \infty$. Such a sequence exists because of our third assumption. We also assume that $|\Omega| = 1$ in order to simplify the notations (so that in particular $f_\Omega u = \int_\Omega u$ for every u). We show, first, that the average $m_n = \int_\Omega u_n$ remains bounded This is obvious if A is the identity, or has a continous inverse. Otherwise we can write (since A1 = 1).

$$\sigma^2 = \int_\Omega |Au_n - g|^2 = \int_\Omega |Au_n - m_n + m_n - g|^2 = \int_\Omega |A(u_n - m_n) + m_n - g|^2$$ so that

$\sigma \geq \|m_n - g\|L^2(\Omega) - \|A(u_n - m_n)\|L^2(\Omega) \geq \|m_n - g -\|L^2(\Omega) - \|A\|\|(u_n - m_n)\|L^2(\Omega)$ where $\|A\|$ denotes the norm of A as a continuous operator of $L^2(\Omega)$. Since N = 1 or 2.

$2 \leq \frac{N}{N-1}$ and by equation (3.1)

(5.1) $$\|u_n - m_n\|L^2(\Omega) = \left\|u_n \quad \int_\Omega u_n(x)dx\right\|L^2(\Omega) \leq c|Du_n|(\Omega)$$

The total variation $|Du_n|(\Omega)$ remains bounded, therefore $m_n = \int_\Omega u_n$ is also bounded. This implies that u_n is bounded in $L^2(\Omega)$ by applying equation (4.1). Upon extracting a subsequence, we may assume that there exist $u \in L^2(\Omega) \cap BV(\Omega)$ such that $u_n \rightarrow u$ weakly in L^2 and $Du_n \rightarrow Du$ weakly --* as a measure. We also have (since A is continuous and linear)

$Au_n \rightarrow Au$. Thus by semi-continuity we get:

$$|Du|(\Omega) \leq \liminf_{n \to \infty} |Du_n|(\Omega) = 1$$

and $$\int_\Omega |Au(x) - g(x)|^2 dx \leq \sigma^2; \int_\Omega Au(x)dx = \int_\Omega g(x)dx$$

we now introduce for $t \in [0,1]$ the function $u^t = tu + (1-t) \int_\Omega g$.

We have for every t, $|Du^t|(\Omega) = |Du|(\Omega) \leq t1 \leq 1$, $\int_\Omega Au^t = \int_\Omega g$, and we have

$$\int |Au^0 - g|^2 = \int_\Omega \left|g - \int_\Omega g\right|^2 \geq \sigma^2 \text{ (by assumption) and}$$

$\int_\Omega |Au^t - g|^2 = \int_\Omega |Au - g|^2 \leq \sigma^2$. By continuity of the map $t \rightarrow \int_\Omega |Au^t - g|^2$, there exists a t_0, [0,1] such that u^{t_0} satisfies equation (4.1) and $|Du^{t_0}|(\Omega) \leq 1$. Necessarily, we must have $|Du^{t_0}|(\Omega) = 1$ so that t_0=1 and u is the solution to the problem (P_1)

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