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UNIQUENESS AND ASYMPTOTIC STABILITY FOR THE RADIAL SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS

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ABSTRACT

Less is known of the uniqueness for the radial solutions $u = u(r)$ of the problem $\Delta u + f(u_+) = 0$ in $R^n\ (n > 2)$, $u(\rho) = 0$, $u'(0) = 0$, besides the cases where $\lim_{r\to\infty} u(r) = 0$; and for the cases based only on the evolution of the functions $f(t)$ and $\frac{d}{dt}\frac{f(t)}{t}$. This paper proves uniqueness for the problem $D_a + f(u_+) = 0\ (r > 0), u(\rho) = 0,\ u'(0) = 0$ based on the assumption that $f \in C^1([0,\infty))$ and that ρ satisfies a boundedness condition. Furthermore, we prove asymptotic stability for $D_a + f(u_+) = 0$ based only on the evolution of $u'(r)$ and $u - \phi(r)f(u)$.

KEY WORDS: Semilinear elliptic equations, Radial solutions, uniqueness, compactness, asymptotic stability

1 INTRODUCTION

For $a = n - 1 > 1\ (n \in N)$ and any $u_0 > 0$, the problem

$$D_a := u'' + \tfrac{a}{r}u' = -f(u_+)\ (r > 0) \tag{1}$$
$$u(0) = u_0 \tag{2}$$
$$u'(0) = 0 \tag{3}$$

is known to have a unique solution $u \in C^2([0,\infty))$ which is positive in some interval $[0,\rho)$, provided $f \in C([0,\infty)) \cap C^1((0,\infty))$ and remains positive, or have a finite number of positive zeros and changes its sign across any of them (Kawano et al., 1988 and Tadie, 1996). This problem arises in many applications (Kwong, 1990). For $\rho > 0$, finite or not, Tadie (1990) investigated some uniqueness conditions for the associated problem

$$D_a + f(u_+) = 0\ (r > 0) \tag{4}$$
$$u(\rho) = u_0 \tag{5}$$
$$u'(0) = 0 \tag{6}$$

based only on the evolution of the functions $f(t)$ and $\frac{d}{dt}\frac{f(t)}{t}$.

Instability (also for parabolic equation) of a radial solution of (1) under the assumption $n \geq 3$, $f(0) = 0$, $f'(0) \leq 0$, $\int_0^\zeta f(t)dt \geq 0$ for some $\zeta \geq 0$, and f subcritical at infinity is proved in Berestycki et al. (1981) and Berestycki and Lions (1983). Cabre' and Capella (2004) established that every nonconstant bounded radial solution u of $-\Delta u = f(u)$ in all R^n is unstable if $n \geq 10$, where f is any C^1 nonlinearity satisfying a generic nondegeneracy condition. Their result applies in particular to every analytic and any power-like nonlinearity. For every $n \geq 11$, and f a polynomial, the authors gave an example of a nonconstant bounded radial stable solution.

In the current analysis, we prove uniqueness for the problem (4) - (6) based on the assumption that $f \in C^1([0,\infty))$ (Kwong and Zhang, 1991), and that ρ satisfies a boundedness condition. Furthermore, we establish asymptotic stability for the general Matukuma equation

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$$D_a + \phi(r) f(u) = 0 \tag{7}$$

based only on the evolution of $u'(r)$ and u ... $r)f(u)$ (1) is the special case that $\phi(r) = 1$. Our proof is simple, concise and independent of the dimension We exemplify our proof by using the example in Cabre and Capella (2004); and disprove the assertion in the paper that stable nonconstant bounded solutions are never radial for $n \leq 2$. We further prove asymptotic stability for the bounded solution of the Matukuma equation for $f(u) = u^3$.

2 UNIQUENESS OF SOLUTION

Lemma 1. Suppose u has a maximum on $[0, \rho]$ If $f \in C^1([0, r])$, then f' has a maximum on $u([0, \rho])$.

Proof Since $[0, \rho]$ is closed and bounded, it is compact. Hence the image $u([0, \rho])$ is compact, and hence closed and bounded. Therefore u has a maximum on $[0, \rho]$ (Lang, 1962). Since $f \in C^1(u[0, \rho])$, it follows that the image $f'(u([0, \rho]))$ is compact; and so closed and bounded. f' has a maximum on $u([0, \rho])$.

Theorem 1 Let $f \in C^1([0, r))$ *and* $\rho < \sqrt{\frac{2(a+1)}{\|f'(u)\|_{C(u[0,r])}}}$. Then the boundary value problem (4) - (6) has at most one solution

Proof. Let u and $\tilde{u}$ be two solutions of (4) - (6). Set $v = u - \tilde{u}$. Then v satisfies

$$v'' + \frac{a}{r} v' = -(f(u) - f(\tilde{u})). \tag{8}$$

Multiplying (8) by r^a gives

$$(r^a v')' = -r^a (f(u) - f(\tilde{u}))$$

which implies that

$$v = -r^{-a} \int_0^r \tau^a \left(f(u(\tau)) - f(\tilde{u}(\tau)) \right) d\tau \tag{9}$$

Therefore

$$v = \int_0^r \sigma^{-a} \int_0^\sigma \tau^a (f(u(\tau)) - f(\tilde{u}(\tau))) \, d\tau \, d\sigma$$
$$= \int_0^r \sigma^{-a} \int_0^\sigma \tau^a \int_{\tilde{u}}^{u} f'(s) \, ds \, d\tau \, d\sigma \tag{10}$$

(10) implies that

$$|v| \leq \| f'(u) \|_{C(u([0,\rho]))} \| u - \tilde{u} \|_{C([0,\rho])} \int_0^r \sigma^{-a} \int_0^\sigma \tau^a \, d\tau \, d\sigma$$
$$\leq \frac{\rho^2 \|f'(u)\|_{C(u[0,\rho])}}{2(a+1)} \|v\|_{C([0,\rho])} \tag{11}$$

where we have used Lemma 1 Maximizing the left side of (11) yields

$$\|v\|_{C[0,\rho]} \leq \frac{\rho^2 \|f'(u)\|_{C(u[0,\rho])}}{2(a+1)} \|v\|_{C[0,\rho]} \tag{12}$$

By the boundedness condition on ρ, (12) implies that $\|v\|_{C([0,\rho])} = 0$ i e $u = \tilde{u}$.

Theorem 2. *(7)* is asymptotically stable if any of the following conditions is satisfied

1. $u - \phi(r)f(u) > 0$ and $u'(r) < 0$
2. $u - \phi(r)f(u) > 0$ and $u'(r) > \frac{r}{a}(u - \phi(r)f(u))$
3. $u - \phi(r)f(u) < 0$ and $u'(r) > 0$
4. $u - \phi(r)f(u) < 0$ and $u'(r) < \frac{r}{a}(u - \phi(r)f(u))$

Proof. Let $x_1 = u$, $x_1' = x_2$. Then (1) is equivalent to the system

$$x_1' = x_2 \tag{13}$$
$$x_2' = -\frac{a}{r}x_2 - \phi(r)f(x_1) \tag{14}$$

We choose as a Lyapunov function, $V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$, which is clearly positive definite. Hence

$$V' = x_1x_1' + x_2x_2'$$
$$= x_1x_2 - \frac{a}{r}x_2^2 - x_2\phi(r)f(x_1) \quad \text{(using (13) and (14))}$$
$$= -\frac{a}{r}x_2[x_2 - \frac{r}{a}(x_1 - \phi(r)f(x_1))] < 0 \tag{15}$$

by each of the conditions 1 - 4. Thus, stability is asymptotic

Theorem 3.

1. For $n < 4(1+r^2) + \frac{9r^2}{4(1+r^2)}$, there exists a polynomial f which admits an asymptotically stable nonconstant bounded radial solution u of (1).

2. The solution $u(r) = \sqrt{3/(1+r^2)}$ for the Matukuma's equation

$$u'' + \frac{2}{r}u' + \frac{1}{1+r^2}u^3 = 0 \tag{16}$$

is asymptotically stable for $r > \sqrt{\sqrt{3}-1}$.

Proof of Theorem 3(1). We consider $u(r) = (1+r^2)^{-\frac{1}{8}}$, a bounded C^2 solution of solution of $-\Delta u = f(u) := ((4n-9)u^9 + 9u^{17})/16$ (Cabre and Capella, 2004). We have

$$u'(r) = -\frac{1}{4}r(1+r^2)^{-\frac{9}{8}} < 0 \tag{17}$$
$$u - f(u) = \frac{1}{(1+r^2)^{\frac{1}{8}}}\left(1 - \frac{4n-9}{16(1+r^2)} - \frac{9}{16(1+r^2)^2}\right) > 0 \tag{18}$$

for $n < 4(1+r^2) + \frac{9r^2}{4(1+r^2)}$. Hence, by the condition 1 of Theorem 2, u is asymptotically stable. Furthermore, since $4(1+r^2) + \frac{9r^2}{4(1+r^2)} > 4$, this result refutes the assertion in Cabre and Capella (2004) that stable nonconstant bounded solutions of (1) are never radial for $n \le 2$.

Proof of Theorem *3(2)*. Here we have

$$u'(r) = \sqrt{3}r(1+r^2)^{-\frac{3}{2}} < 0 \tag{19}$$
$$u - \phi(r)f(u) = \frac{\sqrt{3}}{(1+r^2)^{\frac{1}{2}}}\left(1 - \frac{3}{(1+r^2)^2}\right) > 0 \tag{20}$$

for $r > \sqrt{\sqrt{3}-1}$. Thus, by the condition of Theorem 2, u is asymptotically stable.

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