ON THE SOLUTION OF N-PERSON COOPERATIVE GAMES

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ABSTRACT

In this paper, two existing optimal allocation to N-person cooperative games are reviewed for comparison-The Shapley Value introduced by Shapley (1953) and the Nucleolus introduced by Schmeidler (1969) Given the nonempty Core of an N-person cooperative game, both optimal allocation procedures consider that one point of the Core is more efficient than the other points of the Core while the approaches to choosing the efficient allocation differ Whereas Shapley employed the marginal contribution of the players into the game to achieve his aim, Schmeidler employed the extent of dissatisfaction to achieve his own aim To choose the "best" of the optimal allocations, the Standard error and Coefficient of Variation of solutions were used to discriminate between the two procedures When the two approaches were applied to the same sets of data, the Shapley value method produced smaller standard errors and coefficients of variation than the Nucleolus method The Shapley value approach was therefore chosen as the better one for allocation (the value of the game) to an N-person cooperative game

KEYWORDS: Characteristic Function Coalition, Constant Sum Game Imputation, Player Payoff Strategy The Core

GENERAL BACKGROUND

1.1 INTRODUCTION

Coalition formation is a key problem in automated negotiation among self-interested players, and other multiagent applications A coalition of players can sometimes accomplish things that the individual players cannot, or can accomplish them more efficiently Motivating the players to abide by a solution requires careful analysis only some of the solutions are stable in the sense that no group of players is motivated to break off and form a new coalition This constraint has been studied extensively in cooperative game theory the set of solutions that satisfy it is known as the Core see, for example, Conitzer and Sandholm (2006) González Díaz and Sánchez - Rodríguez (2003), Ferguson (2000), Gomes (1999) and Deng and Papadimitriou (1994) Cooperative game theory considers how to distribute income $v(N)$ generated by a group N to its members The Shapley Value defined by Shapley (1953), is a solution that employs the marginal contribution of the player(s) to select a point in the Core that is stable (more preferable) to all the players in the game see, for example, Littlechild and Owen (1973) González Díaz and Sánchez Rodríguez (2003), Ferguson (2000) and Winston (1987) In general, how well players in a coalition do may depend on what nonmembers of the coalition do see, for example Bernheim et al (1987), Chatterjee et al (1993), Evans (1997), Milgrom and Roberts (1996) Moreno and Wooders (1996), Okada (1996) and Ray (1996) However, in cooperative game theory, coalition formation is usually studied in the context of characteristic function games where the utilities of the coalition members do not depend on the nonmember's action see, for example Kahan and Rapoport (1984), Van der Linden and Verbeek (1985), Zlotkin and Rosenchein (1994), Charnes and Kortanek (1966), Shapley (1967), Wu (1977) One way to interpret this is to consider the coalition member's utilities to be the utilities they can guarantee themselves no matter what the nonmembers do see, for example, Aumann (1959)

The Nucleolus, defined by Schmeidler (1969), is another well-known solution to N-person cooperative game that lexicographically maximizes the sorted vector of excess for all subsets $S \subset N$ More formally, let an imputation $x: N \to R$, represent the income distributed to the members in N with $\sum x_i \quad v(N)$ For each subset $S \subset N$, let $v(S)$ be the revenue generated by the subset S of members, the excess is defined by $e(S, x) = x(S) - v(S)$, where $x(S) = \sum_{i \in S} x_i$ The sorted vector of excess is $e(S_1, x), e(S_2, x), \ldots, e(S_m, x)$, where $m \quad 2^N \quad 2$ such that $e(S_1, x) \leq e(S_2, x), \ldots, \leq e(S_m, x)$ Note that for different imputations x the ordered set $S_1, S_2, \ldots, S_m$ is in general different and the Nucleolus is one of these imputations that maximizes this sorted vectors lexicographically see for example Schmeidler (1969) Solymosi and Raghavan et al (2001)

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1.2 The Philosophy of The Game

The philosophy of an N-person cooperative game is based on the fact that in practice, many players can play the game at the same time and at the end of the game, players gain or suffer losses as the case may be. This type of game includes such games like gambling with a die, business firms engaged in the production of similar products (competition), and/or sharing the profit accruing to their business through cooperation.

A real life situation for this kind of game is given as follows. Suppose there exist **three** players, x_1, x_2 and x_3, say, going for a Christmas outing, visiting their friends and relatives and at each visit a sum of money is given to them to share. At the end of the visit, it is believed that the players must have got a huge sum of money, which will be shared among them equally. In this case, each player will receive the mean allocation $\bar{x} = \dfrac{v(N)}{n}$ where $v(N)$ is the total money generated and n the number of players in the game. Due to rationality assumption inherent in human beings, x_1 may feel he is the brain behind the huge sum of money given to them; that is, th

e three players visited more of x_1 friends and relatives and as such, it was because of his influence that they could get such amount. We can assume that x_1 being a rational thinker will expect that at the end of the visit, he will receive fairly larger amount of money than x_2 and x_3. Similarly, if x_2 feels more important than x_3, x_2 would wish to receive more money than x_3. The question then is how the money will be shared to the three players so that at the end of the day, each player will go home satisfied. The measure of satisfaction here would be quantifiable values, in Nigeria Currency (Naira), say.

To answer this question, we shall consider two solution concepts, the Shapley Value introduced by Shapley (1953) and the Nucleolus introduced by Schmeidler (1969), for comparison. The basis for comparison is the fact that given the Core for an N-person cooperative game with characteristic function, v, one point of the Core is more efficient for allocation (fairer allocation) than other points in the Core. However, the search for the efficient allocation based on the above two solution concepts differs. The standard error and coefficient of variation shall be used to choose the better ("best") optimal allocation. "Best" in this context, refers to the method that gives the minimum standard error and coefficient of variation.

PRELIMINARIES

2.1   Cooperative Game

A cooperative game $\Gamma = (N, v)$ consists of a player set $N = \{1, 2, ..., n\}$ and a value function $v : 2^N \to R$ with $v(\phi) = 0$. The allocation to individual player $i \in N$ is represented by x_i and $x = (x_1, x_2, ..., x_n)$ satisfying $\sum_{i \in N} x_i = v(N)$ is an allocation vector: see, for example, Ferguson (2000).

2.2   The Core

Suppose an imputation, x, is being proposed as a division of the total amount due the players denoted by $v(N)$. If there exists a coalition, S, whose total return from x is less than what that coalition can achieve acting by itself, that is, if $\sum_{i \in s} x_i < v(s)$, then there will be a tendency for the coalition, S, to form a group and reject the proposed x because such a coalition could guarantee each of its members more than they would receive from x. Hence, the Core of a game Γ, denoted by $C(\Gamma)$, is defined as the set of all allocations whose excesses are non-negative. That is, $C(\Gamma) = \{x \in R^n : x(N) = v(N) \ and \ x(S) \geq v(S), \ \forall \ S \subseteq N\}$. As a solution concept, the Core presents a set of imputations without distinguishing one point of the imputation as being preferable to another; see, for example, Conitzer and Sandholm (2006).

In general, the Core of a game can be empty. If the Core is empty, the game is inherently unstable because no matter what outcome is chosen, some subset of agents or coalition is motivated to pull out and form their own coalition. However in this paper, we shall examine games of non-empty Core such that no agents or coalition is motivated to pull out; see, for example, Conitzer and Sandholm (2006).

Again, we shall assume that all allocations in the Core are unbiased estimates of the true value (the mean).

2.3   Characteristic Function

The pair, $G(N, v)$, gives the coalitional form of an N-person cooperative game, where N is the number of players and v is a real-valued function, called the characteristic function of the game, defined on the set, S, of all coalitions (subsets of N), and satisfying

(i) $v(\phi) = 0$, and

(ii) if S and T are disjoint coalitions, $(S \cap T) = \phi$, then $v(S) + v(T) \leq v(S \cup T)$: see, for example, Ferguson (2000).

2.4   Payoff

A quantitative measure or satisfaction a player gets at the end of each play. It is a real–valued function of the outcome of the game; see, for example, Ferguson (2000). A real-valued function here, means quantifiable values, say in Naira

2.5   Imputation

A payoff vector $x = (x_1, x_2, \ldots, x_n)$ of proposed amounts to be received by the players with the understanding

that player i is to receive x_i is called an imputation. Thus, an imputation is an N-vector, $x = (x_1, x_2, \ldots, x_n)$, such that

$x_i \geq v(i)$ for all i and $\sum_{i=1}^{\lambda} x_i = v(N)$ The first condition to expect of an imputation is that no player could be

expected to receive less than what that player could obtain acting alone and a second natural condition to expect of

an imputation, $x = (x_1, x_2, \ldots, x_n)$ is that $x_i > v(i)$ for all players, i Hence an Imputation is said to be individually

rational if $x_i \geq v(i)$ for all i = 1, 2, …N and group rational if $\sum_{i=1}^{\lambda} x_i = v(N)$ see, for example, Ferguson (2000).

2.6   Unstable Imputation

An imputation, x, is said to be unstable through a coalition S, if $v(S) > \sum_{i \in S} x_i$ and we say x is stable

otherwise: see, for example, Ferguson (2000)

2.7   The Shapley Value

Shapley (1953) presented a solution concept to N-person cooperative games called the Shapley value. This, he achieved by computing the value of the ith marginal contribution of the player into the game using the formula

$$x_i \quad \sum_{\substack{S \subset N \\ i \notin S}} P_N(S) \left[v(S \cup i) - v(S) \right] \qquad 1$$

where $P_N(S) \quad \dfrac{|S|!(|N| - |S| - 1)!}{|N|!}$ and other symbols of equation (1) have

the same meanings as highlighted earlier in this work. For any characteristic function, Lloyd Shapley showed that there is a unique reward vector,

$x = (x_1, x_2, \ldots, x_N)$, satisfying the following axioms:

i)   Efficiency $\sum_{i \in N} \phi_i(v) = v(N)$.

ii)   Symmetry if i and j are such that $v(S \cup \{i\}) = v(S \cup \{j\})$ for every coalition, S, not containing i and j, then $\phi_i(v) = \phi_j(v)$.

iii)   Dummy Axiom: if i is such that $v(S) = v(S \cup \{i\})$ for every coalition, S, not containing i, then $\phi_i(v) = 0$.

iv)   Additivity if u and v are characteristic functions, then $\phi(u + v) = \phi(u) + \phi(v)$, where $\phi(v)$ and $\phi(u)$ are functions that assign a value to each of the players in the game

Theorem 1 (Shapley (1953))

Given any N-person cooperative game with characteristic function, v there is a unique reward vector,

$x = (x_1, x_2, \ldots, x_n)$, satisfying axioms i – iv above. The reward to the ith player, (x_i), is given by

$$x_i \quad \sum_{\substack{S \subset N \\ i \notin S}} P_N(S) \left[v(S \cup i) - v(S) \right]$$

where

$$P_N(S) \quad \frac{|S|!(|N| - |S| - 1)!}{|N|!}$$

and |S| is the number of players in S, and for N > 1, N! = N(N – 1) … (2)(1), (0! = 1)

The value $P_N(S)$ has the following interpretation: Suppose that players 1, 2, 3 arrive in a random order, that is, any of the 3! permutations of the random arrival of 3 players

1,	2,	3;	2,	3,	1,
1,	3,	2;	3,	1,	2;
2,	1,	3;	3,	2,	1,

has a 1/3! chance. Suppose that when player i arrives, he finds that the players in the set, S, have already arrived. If player i forms a coalition with the players who are present when he arrives, player i adds v (S∪i) – v (S) to the coalition, S. The probability that when player i arrives, the players in the coalition, S, are present is $P_N(S)$. Using this information, we compute an unbiased and efficient estimate allocation for the three players due to Shapley.

2.7   The Nucleolus

Another interesting value function for an N-person cooperative game is the Nucleolus, a concept introduced by Schmeidler (1969): see, for example, *Gonza'lez – Di'az* and *Sa'nchez - Rodri'guez* (2003) and Carter and Walker (1996). Instead of applying the marginal contribution of the ith player due to Shapley to compute for an unbiased estimate of the value of the game, we look at a given characteristic function, v, and attempt to find an

imputation, $x = (x_1, x_2, ..., x_n)$, that minimizes the worst inequity. That is, we ask each coalition how dissatisfied it is with the proposed imputation or allocation, x, and we attempt to minimize the maximum dissatisfaction: see, for example, Derks and Kuipers (1997), Carter and Walker (1996).

In this work, we shall use some numerical examples to compute for the Shapely value and the Nucleolus and compare the two values, as many of the research work done to compute the Nucleolus were mere theoretical works that has no numerical computation.

3   METHODS AND APPLICATIONS

3.1 Some Applicable Examples

Example 1 (Ferguson (2000)): Consider the three–person game with players 1, 2 and 3, each with two pure strategies and payoff matrices.
If Player 1 chooses strategy a, we have Matrix 1.

Matrix 1

Player 2		Player 3	
		a	b
	a	(a, a, a) = (0, 3, 1)	(a, a, b) = (2, 1, 1)
	b	(a, b, a) = (4, 2, 3)	(a, b, b) = (1, 0, 0)

If Player 1 chooses strategy b, we have Matrix 2.

Matrix 2

Player 2		Player 3	
		a	b
	a	(b, a, a) = (1, 0, 0)	(b, a, b) = (1, 1, 0)
	b	(b, b, a) = (0, 0, 1)	(b, b, b) = (0, 1, 1)

Suppose our interest is in finding the characteristic function of the game denoted by v, we have that v (φ) = 0 since the value of a null coalition denoted by φ is usually zero. It is easy to find the value of the grand coalition v (1, 2, 3) too. This is the largest sum in the eight cells of Matrices 1 and 2 which occurs in cell (a, b, a) of Matrix 1 and which gives the total payoff, v (1, 2, 3) = 9. To find the value of the game for player 1, denoted by v (1), we compute the payoff matrix for the winnings of player 1 against players 2 and 3. This is contained in Matrix 3.

Matrix 3

Player 1		Players 2 and 3			
		(a,a)	(a,b)	(b,a)	(b,b)
	a	0	2	4	1
	b	1	1	0	0

The first and fourth columns dominate the second and third columns of matrix 3 since all the entries of the second and third columns are greater than or equal to the entries of the first and fourth columns. Eliminating the second and third columns, we have the reduced matrix for the players as in Matrix 4.

Matrix 4

	Players 2 and 3	
Player 1	0	1
	1	0

Theorem 2 (Thie (1979))
Consider the game with payoff Matrix 5 below.

Matrix 5

	Column Player	
Row Player	a	b
	c	d

Suppose the game in Matrix 5 has no saddle point. Let $a > b$, $a > c$, $d > b$, $d > c$ or $a < b$, $a < c$,
$d < b$, $d < c$. while $r = (a+d) - (b+c)$, is defined as the weight of the game Then the value of the game. v. is given by

$$v = \frac{(ad - bc)}{r}$$ while the optimal strategies of the game denoted by X_0 and Y_0 are

$$X_0 = \left(\frac{d-c}{r}, \frac{a-b}{r} \right), \quad Y_0 = \left(\frac{d-b}{r}, \frac{a-c}{r} \right)$$

By Theorem 2, we calculate the value of the game for player 1 playing against players 2 and 3 in Matrix 4
above. Thus, $v(1) = \frac{1}{2}$ To find $v(2)$ and $v(3)$, we make similar constructions of matrices for player 2's winnings
against players 1 and 3. and player 3's winnings against players 1 and 2 and find the values of the resulting games
In the matrix of player 2's winnings against players 1 and 3. we observe that there is a saddle point. that is maxmin =
minmax = 0 Hence, $v(2) = 0$. In the matrix of player 3's winnings against players 1 and 2, $v(3)$ is equal to $\frac{3}{4}$

To find $v(1, 3)$. say, we first construct the matrix of the sum of the winnings of players 1 and 3 playing
against player 2 This is done by combining the strategies for players 1 and 3 such that if players 1 and 3 choose
strategy 'a' and player 2 chooses strategy 'a', we add the winnings of players 1and 3 in Matrix 1 together which gives
1. Continuing in this order, we have the Matrix 6 below

Matrix 6

		Player 2	
		a	b
	a,a	1	7
Players	a,b	3	1
1 and 3	b,a	1	1
	b,b	2	1

The lower two rows of Matrix 6 are dominated by the second row; hence, we have the reduced Matrix 7

Matrix 7

	Player 2	
Players 1 and 3	1	7
	3	1

Again, by Theorem 2. we have that $v(1,3) = \frac{5}{2}$

Similarly, we compute the matrix of players 1 and 2 playing against player 3 and the matrix of players 2 and 3
playing against player 1 Both matrices have saddle points and we find that $v(1, 2) = 3$ and $v(2, 3) = 2$ Therefore,
the characteristic function of the game is given by the following array

$$v(1) = \frac{1}{2}, \quad v(1, 2) = 3,$$
$$v(\phi) = 0, \quad v(2) = 0, \quad v(1, 3) = 5/2, \text{ and } v(1, 2, 3) = 9$$
$$v(3) = \frac{3}{4}, \quad v(2, 3) = 2$$

3.2   The Core of a Three-person Cooperative Game

Considering Example 1 the imputations are the points x ($x, x, ..., x$) such that $x + x + x$ 9 and x_1 ½, x 0, x_3 ¾ Plotting these points (x, x, x) in the X, Y, Z coordinates, we have triangle ABC in Figure 1
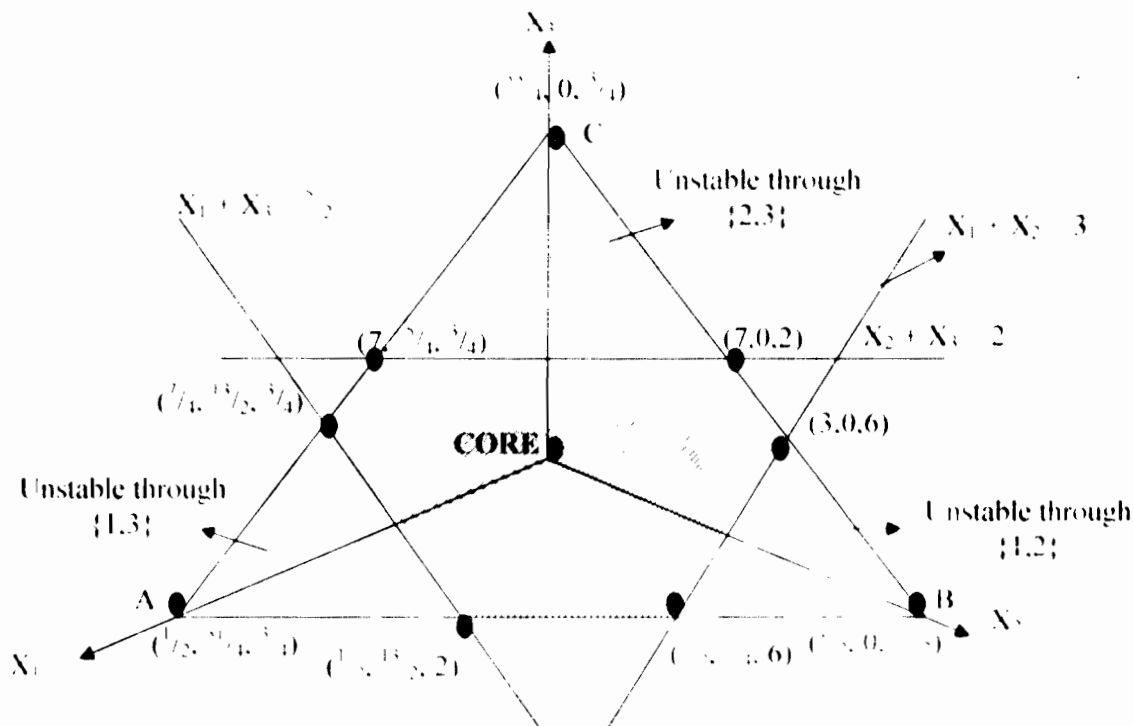

Figure 1: The Core of the game for Example One

In Figure 1 we can also find which imputations are unstable by looking at those Coalitions whose total reward from the imputation are less than what they would get when they play alone. Coalition (2, 3) can guarantee for itself v (2, 3) = 2 bearing in mind the characteristic function of the game, so all points (x_1, x, x_3) with $x_1 + x_3 < 2$ are unstable through Coalition (2, 3)   These are the points below the line $x + x_3$ 2 in Figure 1 Also, Coalition (1, 2) can guarantee for itself v (1, 2) = 3, so that all points below and to the right of the line $x_1 + x_3 = 3$ are unstable Finally,

Coalition (1, 3) can guarantee for itself v (1, 3) = $^5/_2$, such that all points below the line $x_1 + x_3$ $\frac{5}{2}$ are unstable.

The Core is the remaining set of points of triangle ABC in Figure 1 given by the shaded region. This means that at the Core, all the players are satisfied with the allocation (imputation)

3.3   Computation Using the Shapley Method

We shall recall Example 1 above and use the Shapley method to compute an unbiased estimate, x (x_1, x, x_3) ∈ Core

The six different orders of entry of the players are listed along with their respective payoffs in Table 1   In the first row, the players enter in the order 1 2 and 3   Player 1 receives

v (1) = ½ upon entry, then player 2 receives v (1, 2) – v (1) = 3 – ½ = $^5/_2$, finally player 3 receives v (1, 2, 3) – v (1, 2) = 9 – 3 = 6. Each of the six rows is equally likely, each with probability $^1/_6$ The Shapley value of the game is the average of the six numbers in each of the three columns of the players in Table1 and the Table shows the computation of the Shapley value for all the three players simultaneously

Table1: Computation of the Value of the Game for the three players using the Shapley method for Example One

ORDER OF ENTRY	PLAYERS			TOTAL
	1	2	3	
1, 2, 3	½	$2^1/2$	6	9
1, 3, 2	½	$6^1/2$	2	9
2, 1, 3	3	0	6	9
2, 3, 1	7	0	2	9
3, 1, 2	$^7/_4$	$^{13}/_2$	$^3/_4$	9
3, 2, 1	7	$^5/_4$	$^3/_4$	9
AVERAGE	$^1/_6 \times ^{79}/_4$	$^1/_6 \times ^{67}/_4$	$^1/_6 \times ^{35}/_2$	9

Hence, the Shapley value to the game is $x = (x_1, x_2, x_3) = (^{79}/_{24}, ^{67}/_{24}, ^{35}/_{12})$.

3.4 Computation of the Nucleolus

The first step in the computation of the Nucleolus is to choose an imputation, $x \in$ Core, arbitrarily; i.e. $x = (x_1, x_2, x_3) \in$ Core. On the principle that the one who yells loudest gets served first, we look first at those coalitions, S, whose excess, for a fixed allocation, x, is the largest, then we adjust x, if possible to make this largest excess smaller. When the largest excess has been made as small as possible, we concentrate on the next largest excess, and adjust x to make it as small as possible; this will continue in this order (the process of making the largest excess to be small) until no further improvement can be done and finally, we arrive at the Nucleolus.

Again, considering Example 1, to find the Nucleolus of the game, let $x = (x_1, x_2, x_3)$ be an unbiased estimate allocation such that $x_1 + x_2 + x_3 = 9$, and we compute their excesses as found in Table 2.

Table 2: Computation of the Value of the Game for the three players using the Nucleolus method for Example One

The Coalitions (S)	The Value of the Coalitions v (S)	The excess of the Coalitions $e = v$ (S) $- \sum x_i$	The vector of excesses with the first Imputation 3, 3, 3	The vector of excesses with the second imputation 3, $^{11}/_4$, $^{13}/_4$	The Nucleolus $(^{37}/_{12}, ^{31}/_{12}, ^{10}/_3)$
x_1	½	$½ - x_1$	$-^5/_2$	$-^5/_2$	$-^{37}/_{12}$
x_2	0	$- x_2$	-3	$^{11}/_4$	$^{37}/_{17}$
x_3	$^3/_4$	$^3/_4 - x_3$	$-^9/_4$	$^5/_2$	$^{37}/_{12}$
$x_1 x_2$	3	$3 - x_1 - x_2$	-3	$^{11}/_4$	$-^8/_3$
$x_1 x_3$	$^5/_2$	$^5/_2 - x_1 - x_3$	$-^7/_4$	$-^{15}/_4$	$-^{37}/_{12}$
$x_2 x_3$	2	$2 - x_2 - x_3$	-4	-4	$-^{37}/_{12}$

To gain some insight on how to compute the Nucleolus, we first consider an arbitrary point, $x = (3, 3, 3) \in$ Core, say. As seen in Table 2, the vector of excesses is e = $(-^5/_2, -3, -^9/_4, -3, -^7/_4, -4)$. The largest of these excesses is $-^9/_4$ corresponding to coalition x_3. This player will claim that he is being cheated since he has the largest excess of $-^9/_4$. So, some attempt would need to be made to improve things for him by making the Value for x_3 to be larger than it was in the first imputation. Since x_1 has the next largest excess, we keep x_3 constant and decrease x_2 by solving

$$^3/_4 - x_3 = ½ - x_1$$
$$\Rightarrow x_3 - x_1 = ¼ \qquad\qquad (2)$$

Equation 1 implies that the difference between the worth of x_3 and x_1 should be ¼ and this value should be subtracted from x_2 to make the excess of x_3 and x_1 equal. Using this information, will give us the next imputation, $x = (3, ^{11}/_4, ^{13}/_4)$, as in column 5 of Table 2.

At the imputation, $x = (3, ^{11}/_4, ^{13}/_4)$, the next largest excess is at coalitions x_2 and $x_1 x_2$. The best that can be done is to make the excesses of x_1, x_2 and x_3 to be equal so that no player will complain more than the other players and this is done by solving equations 3 and 4, simultaneously

$$\tfrac{1}{2} - x_1 = - x_2 = \tfrac{3}{4} - x_3 \tag{3}$$

$$\text{and } x_1 + x_2 + x_3 = 9 \tag{4}$$

Rearranging equations 3 and 4, we nave

$$\tfrac{1}{2} - x_1 = - x_2 \Rightarrow x_1 - x_2 = \tfrac{1}{2} \tag{5}$$

$$\tfrac{1}{2} - x_1 = \tfrac{3}{4} - x_3 \Rightarrow x_3 - x_1 = \tfrac{1}{4} \tag{6}$$

$$- x_2 = \tfrac{3}{4} - x_3 \Rightarrow x_3 - x_2 = \tfrac{1}{4} \tag{7}$$

$$\text{and } x_1 + x_2 + x_3 = 9 \tag{8}$$

Solving equations 5 to 8 in 3 unknown, simultaneously, gives

$$x_1 = {}^{37}/_{12}, \qquad x_2 = {}^{31}/_{12} \text{ and } x_3 = {}^{10}/_3$$

hence the imputation, $x = (x_1, x_2, x_3) = ({}^{37}/_{12}, {}^{31}/_{12}, {}^{10}/_3)$ is the Nucleolus

Example 2 (Ferguson (2000)): Consider a three-person game with characteristic function v, given by

$$v (1) = 1, \quad v (1, 2) = 4,$$
$$v (\phi) = 0, \qquad v (2) = 0, \quad v (1, 3) = 3, \quad \text{and } v (1, 2, 3) = 8.$$
$$v (3) = 1, \qquad v (2, 3) = 5,$$

The Core of the game is as given in Figure 2 with vertices (7, 0, 1), (1, 6, 1) and (1, 0, 7)
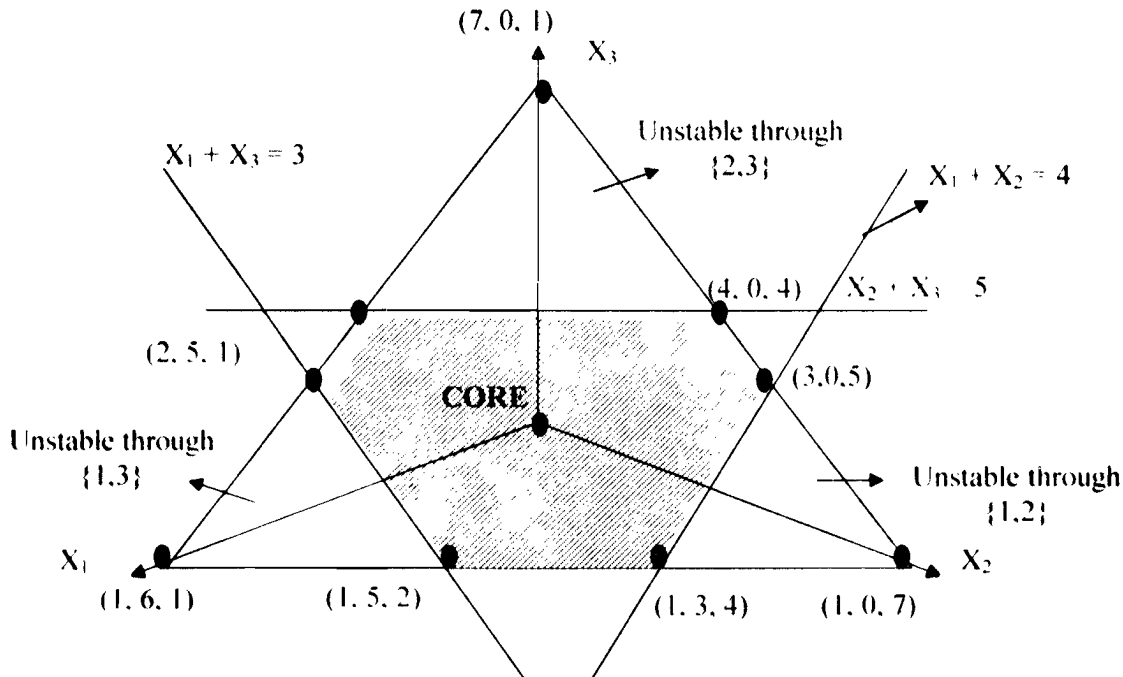

Figure 2: The Core of the game for Example Two

The value of the game using the Shapley method is $x = \left(x_1, x_2, x_3 \right) = ({}^{14}/_6, {}^{17}/_6, {}^{17}/_6) \in$ Core

The value of the game using the Nucleolus method is $x = \left(x_1, x_2, x_3 \right) = ({}^5/_3, {}^{11}/_3, {}^8/_3) \in$ Core

Example 3 (Ferguson (2000)) Consider the three–person game with players 1, 2 and 3, each with two pure strategies and with payoff matrices.
If Player 1 chooses strategy a, we have Matrix 7

Matrix 7

		Player 3	
		a	b
Player 2	a	(a, a, a) = (1, 2, 1)	(a, a, b) = (3, 0, 1)
	b	(a, b, a) = (-1, 6, 3)	(a, b, b) = (3, 2, 1)

If Player 1 chooses strategy b, we have Matrix 8

Matrix 8

Player			Player 3	
			a	b
2		a	(b, a, a) = (-1, 2, 4)	(b, a, b) = (1, 0, 3)
		b	(b, b, a) = (7, 5, 4)	(b, b, b) = (3, 2, 1)

Using similar constructions to those of Example 1, we obtain the characteristic function as follows

$$v\ (1) = {}^1/_3, \quad v\ (1, 2) = 5.$$
$$v\ (\phi) = 0, v\ (2) = 2, \quad v\ (1, 3) = 4, \text{ and } v\ (1, 2, 3) = 16.$$
$$v\ (3) = 1, \ v\ (2, 3) = {}^3/_2.$$

The Shapley of the game is given by $({}^{107}/_{18}, {}^{199}/_{36}, {}^{163}/_{36})$ = (5.94, 5.53, 4.53) while the Nucleolus of the game is given by $({}^{41}/_9, {}^{56}/_9, {}^{47}/_9)$ = (4.56, 6.22, 5.22).

To compare the two optimal allocation methods discussed in this work, we compute their Standard errors and Coefficients of Variation. To do this, we shall employ the idea of variance and minimum mean square error as a property of a good estimator

By an estimator of a parameter θ, we mean a function T of the observations $(x_1,....x_n)$ that is closest to the true value (the mean $\bar{x}$) in some sense. It would, of course, be ideal if there exist a function T such that, compared to any other statistic T', if $E(T-\theta)^2 \le E(T'\ \theta)^2$, where E stands for expectation, that is, the mean square error

(m.s.e) of T is a minimum, then the estimator T is said to be a better estimator than T'. However, estimators satisfying the criterion of minimum mean square error do not generally exist: see, for example, Rao (1973). Hence, we shall use the restriction that the estimators should be unbiased and to choose the best of the estimators, we make use of the fact that an unbiased estimator T with a minimum variance is a better estimator than any other estimator T'. Using their variances, we compute their standard errors and coefficients of variation. The values of the computations are given in Table 3.

Thus, the standard error of the game due to Shapley Value method for Example 1 is 0.1497 while the standard error of the game due to the Nucleolus method for Example 1 is 0.1963. On the other hand, the Coefficient of Variation of the game due to Shapley Value method for Example 1 is 8.64% while it is 11.3% for Example 1 using the Nucleolus method.

Similarly, for Examples 2 and 3, the standard errors of the game due to Shapley Value method are 0.1661 and 0.76 while the Coefficients of Variation of the game for this method are 10.79% and 14.37%, respectively. On the other hand, the standard errors of the game due to Nucleolus are 0.5774 and 0.87 while the Coefficients of Variation of the game for the Nucleolus are 37.5% and 16.32%, respectively

4   COMPARISON OF THE SHAPLEY VALUE AND THE NUCLEOLUS

Table 3: Comparison of the Shapley value and the Nucleolus

No of players	Example 1		Example 2		Example 3	
	The Shapley Value	The Nucleolus	The Shapley Value	The Nucleolus	The Shapley Value	The Nucleolus
X_1	${}^{79}/_{24}$ = 3.29	${}^{37}/_{12}$ = 3.08	${}^{14}/_2$ = 2.33	${}^5/_3$ = 1.6667	${}^{107}/_{18}$ =5.94	${}^{41}/_9$ = 4.56
X_2	${}^{67}/_{24}$ = 2.79	${}^{31}/_{12}$ = 2.58	${}^{17}/_6$ = 2.83	${}^{11}/_3$ = 3.67	${}^{199}/_{36}$ = 5.53	${}^{56}/_9$ = 6.22
X_3	${}^{35}/_{12}$ = 2.92	${}^{10}/_3$ = 3.33	${}^{17}/_6$ = 2.83	${}^8/_3$ = 2.67	${}^{163}/_{36}$ = 4.53	${}^{47}/_9$ = 5.22
Total	9	9	8	8	16	16
Mean	3	3	2.667	2.67	5.33	5.33
Standard Error	0.1497	0.1963	0.1661	0.5774	0.76	0.87
Coefficient of Variation	8.6%	11.3%	10.8%	37.5%	14.37	16.32%

Considering the results for Examples 1, 2 and 3 in Table 3, we notice that the Standard errors for the Shapley value method are less than the Standard errors for the Nucleolus method in the three cases

Hence, the Shapley Value method is considered to be better for efficient allocation than the Nucleolus method and indeed it is the "best" for efficient allocation on the basis of the propositions (minimum standard error and coefficient of variation) stated earlier in this work

5   CONCLUSION

In this paper, two existing optimal solutions to N-person cooperative game, the Shapley Value and the Nucleolus, have been reviewed. Both held the view that given the Core of an N-person cooperative game, one point of

the Core is more efficient for allocation than other points in the Core but the approach of choosing the optimal allocation differs. Whereas Shapley (Shapley Value method) employed the marginal contribution of each of the players into the game to achieve his aim, Schmeidler (Nucleolus method) employed the extent of dissatisfaction of each player, to achieve his aim.

By considering the Standard error and coefficient of variation on a set of randomly generated games, we have been able to discriminate between the two optimal allocations.

Since the standard errors and coefficients of variation using the Shapley value method are less than the standard errors and coefficients of variation using the Nucleolus method, the Shapley Value approach is hereby chosen as the "best" for optimal allocation in N-person cooperative games.

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