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CONTROLLABILITY OF NONLINEAR NEUTRAL VOLTERRA INTEGRODIFFERENTIAL SYSTEMS WITH INFINITE DELAY

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ABSTRACT

We establish a set of sufficient conditions for the controllability of nonlinear neutral Volterra integrodifferential systems with infinite delay. The results are established by using the Schauder fixed point theorem.

KEYWORDS: Controllability, neutral Volterra integrodifferential system, infinite delay, Schauder fixed point theorem.

1. INTRODUCTION

Controllability is one of the fundamental concepts in modern mathematical theory (Klamka, 2002). Controllability is the property of being able to steer between two arbitrary points in the state space using a set of admissible controls (Balachandran and Leelamani, 2006). In the literature there are many definitions of controllability which strongly depend on the class of systems considered (see Klamka, 2002 and Balachandran and Leelamani, 2006).

A neutral differential equation is one in which the derivatives of the past history or derivatives of functionals of the past history are involved as well as the present state of the system (Balachandran et al, 2006). The study of neutral integrodifferential equations with infinite delay has emerged in recent years as an independent branch of modern research because of its connection to many fields such as theoretical epidemiology, physiology, population dynamics and transmission-line theory (Balachandran and Dauer, 1996)

The problem of controllability of nonlinear control processes with infinite delay in finite dimensions has been widely discussed: see Sinha (1985) and Balachandran and Dauer (1996) for systems described by delay differential equations, and Onwuatu (1993) and Umana (2007) for neutral systems.

Controllability of neutral systems with infinite delay in infinite dimensions has also been studied by several authors (see Fu (2003), Balachandran and Anandhi (2004), Bouzahir (2006), and Li et al (2007)). Using the theory of fractional power of operators and the Sadovskii fixed point theorem, Fu (2003) studied the controllability and the local controllability of abstract neutral functional differential systems with unbounded delay. Balachandran and Anandhi (2004) studied the controllability of neutral functional integrodifferential infinite delay systems in Banach spaces by using the analytic semigroup theory and the Nuusbaum fixed point theorem whereas Bouzahir (2006) derived a set of sufficient conditions for the controllability of neutral functional differential equations with infinite delay using the integrated semigroups theory. Recently, with the help of Sadovskii's fixed point theorem, Li et al (2007) studied the controllability of abstract neutral functional integrodifferential systems with infinite delay.

The aim of this paper is to study the controllability of the perturbed neutral Volterra integrodifferential system with infinite delay on a finite interval $J = [0, t_1]$, $t_1 > 0$, when its unperturbed linear system is assumed controllable. Our approach, similar to one used by Do (1990) for nonlinear neutral systems, is to define the appropriate control and its corresponding solution by an integral equation. This equation is then solved by applying the Schauder fixed point theorem.

2. PRELIMINARIES

Let Q denote the Banach space of all continuous functions

$$(x, u) : J \times J \to R^n \times R^m$$

with the norm defined by

$$\|(x, u)\| = \|x\| + \|u\|$$

where $\|x\| = \sup|x(t)|$ for $t \in [0, t_1]$ and $\|u\| = \sup|u(t)|$ for $t \in [0, t_1]$.

We shall consider the following linear neutral Volterra integrodifferential systems with infinite delay represented by

$$\frac{d}{dt}\left[x(t) - \int_{-\infty}^{t} C(t-s)x(s)ds - g(t)\right] = A(t)x(t) + \int_{-\infty}^{t} G(t-s)x(s)ds + B(t)u(t) \qquad (2.1)$$

and the nonlinear system

$$\frac{d}{dt}\left[x(t) - \int_{-\infty}^{t} C(t-s)x(s)ds - g(t)\right] = A(t)x(t) + \int_{-\infty}^{t} G(t-s)x(s)ds$$

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$$+B(t)u(t)+f\big(t,x(t),u(t)\big) \qquad (2.2)$$

$x(t)=\phi(t)$ on $(-\infty,0)$ where the initial function ϕ is continuous and bounded on R^n and where $x\in R^n$, $u\in R^m$, and $B(t)$ is a continuous $n\times m$ matrix valued function. The $n\times n$ matrices $A(t)$, $C(t)$ and $G(t)$ are continuous in their arguments. The $n-$vector functions f and g are respectively continuous and absolutely continuous.

Equivalently, systems (2.1) and (2.2) take respectively the forms

$$\frac{d}{dt}\left[x(t)-\int_0^t C(t-s)x(s)ds-g(t)-\int_{-r}^0 C(t-s)\phi(s)ds\right]=A(t)x(t)+\int_0^t G(t-s)x(s)ds$$
$$+\int_{-r}^0 G(t-s)\phi(s)ds+B(t)u(t) \qquad (2.3)$$

and

$$\frac{d}{dt}\left[x(t)-\int_0^t C(t-s)x(s)ds-g(t)-\int_{-r}^0 C(t-s)\phi(s)ds\right]=A(t)x(t)+\int_0^t G(t-s)x(s)ds$$
$$+\int_{-r}^0 G(t-s)\phi(s)ds+B(t)u(t)+f\big(t,x(t),u(t)\big) \qquad (2.4)$$

Following Balachandran (1992) we give the following definition:

Definition 2.1: The system (2.1) or (2.2) is said to be controllable on J if for each initial function $\phi\in C_n(-\infty,0]$ and for every $x_1\in R^n$, there exists a control $u(t)$, defined on J, such that the solution of the system (2.1) or (2.2) satisfies $x(t_1)=x_1$

The solution of (2.3) can be written, as in Wu (1988), in the form

$$x(t)=Z(t)\left[x(0)-g(0)-\int_{-r}^0 C(-s)\phi(s)ds\right]+g(t)+\int_{-\infty}^0 C(t-s)\phi(s)ds$$
$$+\int_0^t \dot Z(t-s)\left[g(s)+\int_{-r}^0 C(s-\tau)\phi(\tau)d\tau\right]ds$$
$$+\int_0^t Z(t-s)\left[\int_{-\infty}^0 G(s-\tau)\phi(\tau)d\tau\right]ds+\int_0^t Z(t-s)B(s)u(s)ds,$$

where $\dot Z(t-s)=\partial/\partial t\, Z(t-s)$ and $Z(t)$ is an $n\times n$ continuously differentiable matrix satisfying the equation

$$\frac{d}{dt}\left[Z(t)-\int_0^t C(t-s)Z(s)ds-\int_{-r}^0 C(t-s)Z(s)ds\right]=A(s)Z(t)+\int_0^t G(t-s)Z(s)ds$$
$$+\int_{-r}^0 G(t-s)Z(s)ds$$

with $Z(0)=I$ and the solution of the nonlinear system (2.4) is given by

$$x(t)=Z(t)\left[x(0)-g(0)-\int_{-\infty}^0 C(-s)\phi(s)ds\right]+g(t)+\int_{-r}^0 C(t-s)\phi(s)ds$$
$$+\int_0^t \dot Z(t-s)\left[g(s)+\int_{-r}^0 C(s-\tau)\phi(\tau)d\tau\right]ds$$
$$+\int_0^t Z(t-s)\left[\int_{-r}^0 G(s-\tau)\phi(\tau)d\tau\right]ds$$
$$+\int_0^t Z(t-s)\left[B(s)u(s)+f\big(s,x(s),u(s)\big)\right]ds$$

Define the controllability matrix W by

$$W(0,t)=\int_0^t Z(t-s)B(s)B^T(s)Z^T(t-s)ds$$

where T denotes the matrix transpose.

Proposition 2.1: The system (2.1) is controllable on J if and only if W is nonsingular

Proof: This is equivalent to Theorem 1 of Balachandran (1992).

It is clear that x_1 can be obtained if there exist continuous functions $x(\cdot)$ and $u(\cdot)$ such that

$$u(t)=B^T(t)Z^T(t_1-t)W^{-1}(0,t_1)\Big[x_1-Z(t_1)\Big(x(0)-g(0)-\int_{-r}^0 C(-s)\phi(s)ds\Big)$$

$$g(t_1)-\int_0^{t_1} C(t_1-s)g(s)ds-\int_0^{t_1}\dot{Z}(t_1-s)\left[g(s)+\int_{-\infty}^0 C(s-\tau)\phi(\tau)d\tau\right]ds$$
$$-\int_0^{t_1} Z(t_1-s)\left[\int_{-\infty}^0 G(s-\tau)\phi(\tau)d\tau\right]ds-\int_0^{t_1} Z(t_1-s)f(s,x(s),u(s))ds\Bigg] \qquad (2.5)$$

and

$$x(t)=Z(t)\left[x(0)-g(0)-\int_{-\infty}^0 C(-s)\phi(s)ds\right]+g(t)+\int_{-\infty}^0 C(t-s)\phi(s)ds$$
$$+\int_0^t \dot{Z}(t-s)\left[g(s)+\int_{-\infty}^0 C(s-\tau)\phi(\tau)d\tau\right]ds+\int_0^t Z(t-s)\left[\int_{-\infty}^0 G(s-\tau)\phi(\tau)d\tau\right]ds$$
$$+\int_0^t Z(t-s)\left[B(s)u(s)+f(s,x(s),u(s))\right]ds. \qquad (2.6)$$

We now prove that such x and u exist.

3. Main Results

Theorem 3.1: In (2.2) assume that the continuous functions $F_i : R^n \times R^m \to R^+$ and L^1 functions $\alpha_i : J \to R^+$, $i=1,2,\dots,q$ are such that

$$|f(t,x,u)|\le\sum_{i=1}^q \alpha_i(t)F_i(x,u) \quad \text{for every } (t,x,u)\in J\times R^n\times R^m$$

where

$$\limsup_{r\to\infty}\left(r-\sum_{i=1}^q c_i \sup\{F_i(x,u):\|(x,u)\|\le r\}\right)=+\infty. \qquad (3.1)$$

Then the controllability of (2.1) implies the controllability of (2.2) on J.

Proof:

Define $T:Q\to Q$ by

$$T(x,u)=(y,v),$$

where

$$v(t)=B^T Z^T(t_1-t)W^{-1}(0,t_1)\left[x_1-Z(t_1)\left(x(0)-g(0)-\int_{-\infty}^0 C(-s)\phi(s)ds\right)\right.$$
$$-g(t_1)-\int_0^{t_1} C(t_1-s)g(s)ds-\int_0^{t_1}\dot{Z}(t_1-s)\left[g(s)+\int_{-\infty}^0 C(s-\tau)\phi(\tau)d\tau\right]ds$$
$$\left.-\int_0^{t_1} Z(t_1-s)\left[\int_{-\infty}^0 G(s-\tau)\phi(\tau)d\tau\right]ds-\int_0^{t_1} Z(t_1-s)f(s,x(s),u(s))ds\right] \qquad (3.2)$$

and

$$y(t)=Z(t)\left[x(0)-g(0)-\int_{-\infty}^0 C(-s)\phi(s)ds\right]+g(t)+\int_{-\infty}^0 C(t-s)\phi(s)ds$$
$$+\int_0^t \dot{Z}(t-s)\left[g(s)+\int_{-\infty}^0 C(s-\tau)\phi(\tau)d\tau\right]ds+\int_0^t Z(t-s)\left[\int_{-\infty}^0 G(s-\tau)\phi(\tau)d\tau\right]ds$$
$$+\int_0^t Z(t-s)\left[B(s)u(s)+f(s,x(s),u(s))\right]ds \qquad (3.3)$$

By our assumptions, the operator T is continuous. Clearly the solutions u and x to (2.5) and (2.6) are fixed points of T. Our immediate aim now is to establish the existence of such fixed points by using the Schauder fixed point theorem. Indeed, let

$$F_i(r)=\sup\{F_i(x,u):\|(x,u)\|\le r_0\}.$$

Since the growth condition in (3.1) is valid, there exists a constant $r_0>0$ such that

$$r_0-\sum_{i=1}^q c_i F_i(r_0)\ge d$$

or

$$\sum_{i=1}^q c_i F_i(r_0)+d\le r_0$$

for some d. Now introduce the following notations

$$K=\max\{\|Z(t-s)\|:0\le s\le t\le t_1\},$$

$$k = \max\{\|Z(t-s)B(s)\|t_1, 1\}.$$

$$\|\alpha_i\| = \int_0^{t_1} |\alpha_i(s)| ds, \quad (i = 1, 2, \ldots, q),$$

$$a_i = 3k\left\{\|B^T(s)Z^T(t_1 - s)\| \|W^{-1}(0, t_1)\| \|Z(t_1 - s)\| \|\alpha_i\|\right\}, \quad (i = 1, 2, \ldots, q),$$

$$b_i = 3K\|\alpha_i\|, \quad (i = 1, 2, \ldots, q),$$

$$c_i = \max\{a_i, b_i\}, \quad (i = 1, 2, \ldots, q).$$

$$d_1 = 3k\|B^T(s)Z^T(t_1 - s)\| \|W^{-1}(0, t_1)\| \Big[|x_1| + \|Z(t_1)\| \Big| x(0) - g(0) - \int_{-r}^{0} C(-s)\phi(s)ds \Big|$$

$$+ |g(t_1)| + \Big| \int_{-r}^{0} C(t_1 - s)\phi(s)ds \Big| + \int_0^{t_1} \|\dot{Z}(t_1 - s)\| \Big[g(s) + \int_{-r}^{0} C(s-\tau)\phi(\tau)d\tau \Big] ds$$

$$+ \int_0^{t_1} \|Z(t_1 - s)\| \Big[\int_{-r}^{0} G(s-\tau)\phi(\tau)d\tau \Big] ds \Big],$$

$$d_2 = 3\|Z(t_1)\| \Big| x(0) - g(0) - \int_{-r}^{0} C(-s)\phi(s)ds \Big| + |g(t_1)| + \Big| \int_{-r}^{0} C(t_1 - s)\phi(s)ds \Big|$$

$$+ \int_0^{t_1} \|\dot{Z}(t_1 - s)\| \Big[g(s) + \int_{-r}^{0} C(s-\tau)\phi(\tau)d\tau \Big] ds$$

$$+ \int_0^{t_1} \|Z(t_1 - s)\| \Big[\int_{-r}^{0} G(s-\tau)\phi(\tau)d\tau \Big] ds,$$

$$d = \max\{d_1, d_2\}.$$

Now let

$$Q_{r_0} = \{(x, u) \in Q : \|(x, u)\| \le r_0\}$$

If $(x, u) \in Q_{r_0}$, then from (3.2) and (3.3) we have

$$\|v\| = \|B^T(t)Z^T(t_1 - t)\| \|W^{-1}(0, t_1)\| \Big[|x_1| + \|Z(t_1)\| \Big| x(0) - g(0) - \int_{-r}^{0} C(-s)\phi(s)ds \Big|$$

$$+ |g(t_1)| + \Big| \int_{-r}^{0} C(t_1 - s)\phi(s)ds \Big| + \int_0^{t_1} \|\dot{Z}(t_1 - s)\| \Big[g(s) + \int_{-r}^{0} C(s-\tau)\phi(\tau)d\tau \Big] ds$$

$$+ \int_0^{t_1} \|Z(t_1 - s)\| \Big[\int_{-r}^{0} G(s-\tau)\phi(\tau)d\tau \Big] ds + \int_0^{t_1} \|Z(t_1 - s)\| \sum_{i=1}^{q} \alpha_i(s)F_i(x(s), u(s))ds \Big]$$

$$\le \frac{d_1}{3k} + \sum_{i=1}^{q} \frac{1}{3k}\alpha_i F_i(r_0) \le \frac{1}{3k}\Big(d + \sum_{i=1}^{q} c_i F_i(r_0) \Big)$$

$$\le \frac{1}{3k} r_0 < \frac{r_0}{3}$$

and

$$\|y\| = \|Z(t)\| \Big| x(0) - g(0) - \int_{-r}^{0} C(-s)\phi(s)ds \Big| + |g(t)| + \Big| \int_{-r}^{0} C(t-s)\phi(s)ds \Big|$$

$$+ \int_0^{t} \|\dot{Z}(t-s)\| \Big[g(s) + \int_{-r}^{0} C(s-\tau)\phi(\tau)d\tau \Big] ds + \int_0^{t} \|Z(t-s)\| \Big[\int_{-r}^{0} G(s-\tau)\phi(\tau)d\tau \Big] ds$$

$$+ \int_0^{t} \|Z(t-s)B(s)\| \|v\| ds + \int_0^{t} \|Z(t-s)\| \sum_{i=1}^{q} \alpha_i(s)F_i(x(s), u(s))ds$$

$$\le \frac{d}{3} + k\|v\| + K\sum_{i=1}^{q} \|\alpha_i\| F_i(r_0) \le \frac{d}{3} + k\|v\| + \sum_{i=1}^{q} \frac{1}{3} c_i F_i(r_0)$$

$$\le \frac{1}{3}\Big(d + \sum_{i=1}^{q} c_i F_i(r_0) \Big) + k\|v\| \le \frac{r_0}{3} + \frac{r_0}{3} = 2\Big(\frac{r_0}{3}\Big)$$

$$\le \frac{r_0}{3} + \frac{r_0}{3} = 2\Big(\frac{r_0}{3}\Big)$$

Hence T maps Q_{r_0} into itself. Further, it is easy to see that $T(Q_r)$ is equicontinuous for all $r > 0$ (Do, 1990). By the application of Arzela-Ascoli theorem, $T(Q_{r_0})$ is compact in Q. Since the set Q is bounded, closed, and convex, then by Schauder's fixed point theorem, T has a fixed point $(x,u) \in Q$ such that $T(x,u) = (x,u)$. Hence, for $(y,v) = (x,u)$, we have

$$x(t) = Z(t)\left[x(0) - g(0) - \int_{-r}^{0} C(-s)\phi(s)ds\right] + g(t) + \int_{-r}^{0} C(t-s)\phi(s)ds$$
$$+\int_0^t \dot{Z}(t-s)\left[g(s) + \int_{-r}^{0} C(s-\tau)\phi(\tau)d\tau\right]ds + \int_0^t Z(t-s)\left[\int_{-r}^{0} G(s-\tau)\phi(\tau)d\tau\right]ds$$
$$+\int_0^t Z(t-s)\left[B(s)u(s) + f\left(s, x(s), u(s)\right)\right]ds.$$

Thus the solutions of (2.5) and (2.6) exist. Hence the system (2.2) is controllable on J.
Inspired by these ideas we have the following corollaries.

Corollary 3.1: For the system (2.2) assume that the continuous function f satisfies the condition

$$\lim_{\|(x,u)\|\to\infty} \frac{|f(t,x,u)|}{\|(x,u)\|} = 0 \qquad (3.4)$$

uniformly in $t \in J$. Then the controllability of (2.1) implies the controllability of (2.2) on J.

Proof: Let

$$F(x,u) = \sup\left\{|f(t,x,u)| : t \in J\right\}.$$

Then

$$\lim_{r\to\infty}\left(r - \sum_{i=1}^{q} c_i \sup\left\{F_i(x,u) : |(x,u)| \le r\right\}\right) = +\infty \qquad (3.5)$$

if

$$\liminf_{r\to\infty}\left(\frac{1}{r}\right)\sup\left\{F(x,u) : |(x,u)| \le r\right\} < \frac{1}{c_1}. \qquad (3.6)$$

But condition (3.4) implies (3.6) by a modification of an argument of Do (1990). Therefore (3.5) is valid and Theorem 3.1 can be concluded.

Corollary 3.2: for the system (2.2) assume

(i) $f : J \times R^n \times R^m$ is locally bounded in u, that is, for any $M > 0$, there exists an $L > 0$ such that $\|f(t,x,u)\| < L$ for all $(t,x) \in J \times R^n$ and for all $\|u\| \le M$.

(ii) $$\lim_{\|u\|\to\infty} \frac{\|f(t,x,u)\|}{\|u\|} = 0 \qquad (3.7)$$

uniformly in $(t,x) \in J \times R^n$. Then the controllability of (2.1) implies the controllability of (2.2) on J

Proof: Let $F(x,u) = \sup\left\{\|f(t,x,u)\| : t \in J\right\}$ Then

$$\|f(t,x,u)\| \le F(x,u) \quad \text{for every } (t,x,u) \in J \times R^n \times R^m$$

Thus

$$\lim_{r\to\infty}\left(\frac{1}{r}\right)\sup\left\{F(x,u) : \|(x,u)\| \le r\right\} = 0$$

is valid because of (3.7) and as a consequence, (3.6) holds and the results follows.

CONCLUSION

We have formulated and proved sufficient conditions for the controllability of nonlinear neutral Volterra integrodifferential systems with infinite delay in a given finite time interval. The approach used here has been to define the appropriate control and its corresponding solution by an integral equation. This equation was then solved using the well known Schauder fixed point theorem.

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