ANALYSING STAGECOACH NETWORK PROBLEM USING DYNAMIC PROGRAMMING ALGORITHM.

P. O. EKOKO

ABSTRACT

The stagecoach problem is a special type of network analysis problem in which the cities (nodes) are arranged in stages. By such human or natural arrangement, a journey from City 1 in stage 1 to City n in stage n involves visiting only one city in each intermediate stage. The stagecoach problem involves the determination of the minimum cost flow in which basically two methods have been in use. One of the methods is by listing all possible routes, computing their corresponding costs and selecting the minimum cost route. This procedure entails much computational effort. The second procedure is to determine the suboptimal cost of routes between any two consecutive stages. Though this second approach has less computational effort, its overall optimal policy could be wrong because the choice of a least expensive route at one stage may result in our adopting a more expensive route at a later stage. In this paper we present a recursive dynamic programming algorithm for solving the stagecoach problem. The algorithm is computationally more efficient than the first method as it obtains its minimum total cost using the suboptimal policies of the different stages without computing the cost of all the routes. By the dynamic programming algorithm of this paper, the possibility of missing the minimum cost route is ruled out as could happen in the second approach. The dynamic programming algorithm for obtaining the minimum cost path in a stagecoach problem is numerically illustrated.

KEYWORDS: Network analysis, stagecoach, dynamic programming algorithm.

1. INTRODUCTION

Taylor (1996) defined a network as an arrangement of paths connected at various points through which one or more items move from one point to another in such a way which according to Marion (1991) meets definite goals By the network technique, mathematical models can be used to graphically resent a project in the form of a network, Okpamen (2001). The most common types include networks of highway systems, telephone networks and railroad systems, Weiss and Gershon (1993). A network representation provides a powerful visual and conceptual aid. Anderson et al (1997), which as contained in Hillier and Lieberman (2001) are used to portray the relationship between the components of systems in virtually every field of scientific. social and economic endeavour One of the most exciting developments in Operations Research in recent years has been the unusually rapid advance in both the methodology and application of network optimization models, Pilcher (1992)

Five important network problems and some basic procedures of solving them have so far been identified The first three types of the problems are the shortest-route problem, the minimum-spanning tree problem and the maximum-flow problem, which are discussed in Render and Stair (1997). The fourth Type is the minimum cost flow problem and the last is project planning and control with Project Evaluation and Review Technique (PERT) and Critical Path Method (CPM) A special case of the fourth type of the network problems is the consideration in this paper This paper considers a network problem, which has an origin and a destination with cities in between them that are arranged in stages Wagner (1989) refers to this type of minimum cost flow problem as a stagecoach network problem. In this paper we have modeled the stagecoach problem using dynamic programming and an algorithm has been developed for solving the stagecoach problem

Ebert and Adam (1978) stated that though the schematic models. PERT and CPM are the most common and useful methods of analysing a project, the Gantt Chart also known as the Bar Chart was originally developed for industrial production management. However. Nunnally (1980) identified the failure of the Gantt Chart to show the relationships between project activities and its inability to show the effect of delay or change in one activity on the project as some of its disadvantages.

P. O. Ekoko, Department of Mathematics University of Benin Benin City. Edo State. Nigeria

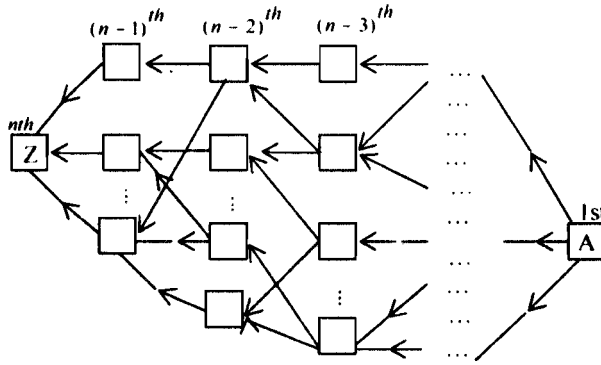## 2. Formulation of the Dynamic Programming Model

Fig.1: Showing stagecoach Network Diagram Having n Stages.

The network diagram in fig. 1 is a representation of the general stagecoach problem having n stages. The journey is to be made in stages from the origin which is city A to the destination which is city Z. The cities in the geographical area are found to be arranged in different stages by some natural groupings possibly based on landscape outlook, rivers, languages, weather e.t.c. Apart from the origin and destination each city in each stage is connected to one or more neighboring cities situated in the stages to its right and left. And the cost of moving from one city to another neighboring city where there is route is known. The network problem is to determine the minimum total cost of travelling from the origin to the final destination. As mentioned in section 1, there are many methods of solving these kinds of network problems. These methods include the PERT and CPM. But in this paper we have developed an algorithm based on dynamic programming for solving the stagecoach problem.

Dynamic programming is a relatively efficient enumeration technique that has a computational scheme, which is based on the principle of optimality. The optimality principle implies a sequential decision process in which a problem involving several variables is broken down into a sequence of simpler problems each involving a single variable, with prior decisions not affected by subsequent decisions. Each component problem can then be solved by the best available procedure, usually enumeration. A common feature in dynamic programs is that the problem being considered can be divided into different stages or subproblems with a decision required at each stage. We now continue by defining some mathematical symbols used in the rest of the paper.

At stage $(n-r)$ of an n stage network problem: Let S_{n-r} be a city in stage $(n-r)$ and $C_{s_{n-r}, j_{n-r+1}}$ be the cost of travelling from city s_{n-r} to city j_{n-r+1} which is at stage $(n-r+1)$. Note that at stage $(n-r)$, the traveler is r stages away from his destination. Also let $C_{n-r}\left(s_{n-r}^{*}\right)$ denote the minimum cost when the traveler moves from city s_{n-r} (in state $(n-r)$) to city j_{n-r+1} in state $(n-r+1)$). Then we define

$$C_{n-r}\left(s_{n-r}^{*}\right) = \min_{j_{n-r+1}} \left\{ c_{s_{n-r}, j_{n-r+1}} + c_{n-r+1}\left(j_{n-r+1}\right) \right\} \tag{1}$$

for $j_{n-r+1} = 1_{n-r+1}, 2_{n-r+1}, \cdots j_{n-r+1}^{*}$

$$C_{n-r}\left(s_{n-r}^{*}\right) = c_{s_{n-r}, j_{n-r+1}} + c_{n-r+1}\left(j_{n-r+1}^{*}\right) \tag{2}$$

Where j_{n-r+1}' is the number of cities in stage $(n-r+1)$ that connect city s_{n-r} of stage $(n-r)$. j_{n-r+1}' depends on the network flow. Alternatively for each s_{n-r} in stage $(n-r)$,

$$C_{n-r}\left(s_{n-r}^{*}\right) = \min \left\{ c_{s_{n-r}, 1_{n-r+1}} + c_{n-r+1}\left(1_{n-r+1}\right), c_{s_{n-r}, 2_{n-r+1}} + c_{n-r+1}\left(2_{n-r+1}\right), \cdots; c_{s_{n-r}, j_{n-r+1}} + c_{n-r+1}\left(j_{n-r+1}'\right) \right\} \tag{3}$$

Equation (3) is done for

$$S_{n-r} = \left\{ 1_{n-r}, 2_{n-r}, \cdots; s_{n-r}' \right\}$$

i.e. done for s' times in stage $(n-r)$. The results of the computations using the s' cities in stage $(n-r)$ and j' cities in the next $(n-r+1)th$ stage obtained from all the s'_{n-r} cities can be tabulated in table 1

Table 1: Showing the Computations of stage $(n-r)$

s_{n-r} \ j_{n-r+1} (To)	1_{n-r+1} 2_{n-r+1} ··· j'_{n-r+1}	$c_{n-r}(s^*_{n-r})$	$d_{n-r}(s^*_{n-r})$
(From) 1_{n-r} 2_{n-r} ⋮ s'_{n-r}	B	C	D

In table 1, each element position in B is computed using $\{c_{s_{n-r},j_{n-r+1}} + c_{n-r+1}(j_{n-r+1})\}$ while entries in C are obtained using equation (1) and D are the decisions corresponding to entries in C.

One important notation for each stage $(n-r)$ is the suboptimal decision denoted by $d_{n-r}(s^*_{n-r})$ to be taken at city s_{n-r}. This suboptimal decision at stage $(n-r)$ is the city j^*_{n-r+1} at the $(n-r+1)th$ stage that yields the suboptimal cost $c_{n-r}(s^*_{n-r})$ and is given as:

$$d_{n-r}(s^*_{n-r}) = j^*_{n-r+1} \tag{4}$$

A peculiar note should be taken of what happens at the last stage which is the nth stage. At the nth stage in fig. 1, when $r=0$, the destination is city Z and applying the recursive equation (2), we have

$$c_n(s^*_n) = 0 \tag{5}$$

This is because the journey terminates at city Z which is at the nth stage and obviously the corresponding suboptimal decision to be made is STOP. That is

$$d_n(s^*_n) = \text{STOP} \tag{6}$$

Also, if the traveler has 3 more stages to go through, i.e. when $r=3$, equation (2) becomes.

$$c_{n-3}(s^*_{n-3}) = \min_{j_{n-2}} c_{s_{n-3},j_{n-2}} + c_{n-2}(j_{n-2}) = c_{s_{n-3},j^*_{n-2}} + c_{n-2}(j^*_{n-2}) \tag{7}$$

And, $$d_{n-3}(s^*_{n-3}) = j^*_{n-2} \tag{8}$$

We can now express the dynamic programming model in algorithmic form.

The Dynamic Programming Algorithm
Step 1: nth stage Computation
Since this is the last stage, the suboptimal minimum cost can be determined from equation (5) and the corresponding optimal decision is to stop as expressed by equation (6). That is

$$c_n(s^*_n) = 0 \quad ,$$

and $$d_n(s^*_n) = \text{STOP}$$

Step 2: $(n-r)^{th}$ stage computation
For each s_{n-r} in stage $(n-r)$, we apply equation (3) as follows:

$$c_{n-r}(s^*_{n-r}) = \min\{c_{s_{n-r},1_{n-r+1}} + c_{n-r+1}(1_{n-r+1}), c_{s_{n-r},2_{n-r+1}} + c_{n-r+1}(2_{n-r+1}), \cdots, c_{s_{n-r},j_{n-r+1}} + c_{n-r+1}(j'_{n-r+1})\}$$

$$= c_{s_{n-r},j^*_{n-r+1}} + c_{n-r+1}(j^*_{n-r+1})$$

And $d_{n-r}(s_{n-r}) = j^*_{n-r+1}$ is the corresponding optimal decision. Based on the routes connecting cities in stage $(n-r)$ to cities in stage $(n-r+1)$, we complete Table 1 for stage $(n-r)$. In a similar manner, the table based on routes connecting stage $(n-r-1)$ cities and stage $(n-r)$ cities is completed. This continues until $r = n-1$ when we determine

$$c_1\left(s_1^*\right) = c_{s_1, j_2^*} + c_2\left(j_2^*\right)$$

(9)

$d_1\left(s_1^*\right) = j_2^*$ is the corresponding suboptimal decision of stage 1

Stage 3: Determination of the overall minimum total policy cost.

$c_1\left(s_1^*\right)$ in equation (9) which is computed in step 2 gives the overall minimum total policy cost (i.e. overall optimal cost) of the network problem. By a recursive process the corresponding optimal route through the network is obtained from the various suboptimal decisions as follows:

Start from $\boxed{j_1^*} \longrightarrow \boxed{j_2^*} \longrightarrow \boxed{j_3^*} \rightarrow \boxed{j_{n-r}^*} \rightarrow \;\; \dots \rightarrow \;\;\;\; \rightarrow \boxed{j_n^*}$

Each of the n stages of the network produces one of these $j_\bullet^*$ cities which are determined by $d_\bullet\left(s_\bullet^*\right)$ suboptimal decisions. Note that while the city $j_1^* = A$ and the suboptimal decision is START, the city $j_n^* = Z$, and the corresponding suboptimal decision is STOP. We now illustrate the algorithm using the following example.

3.   NUMERICAL ILLUSTRATION

Example

A journey is to be made from city 1 across many cities in stages to city 10 which is the destination. The network flow diagram is shown in fig. 2. The cost of transportation (in hundred of naira) along each route (between two cities) is indicated along each route in the diagram. Determine the cities along the route that give the minimum total cost from city 1 to city 10 and specify the minimum total cost.
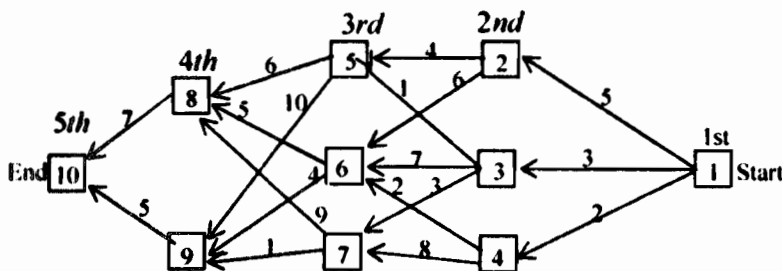
Fig. 2: Showing Network Flow Having 5 Stages.

Solution

The network diagram has 5 stages i.e. $n = 5$. By step 1,

$c_5\left(s_5^*\right) = c_5(10) = 0 \; and \; d_5\left(s_5^*\right) = $ STOP.

By step 2, when $r = 1$, we determine

$c_4\left(s_4^*\right)$ for $s_4 = \{8, 9\}$

i.e. $c_4(8) = \{c_{8,10} + c_5(10)\} = 7$

$c_4(9) = \{c_{9,10} + c_5(10)\} = 5$

The results are summarised in the following table.

Table for 4th stage

		To 10	$c_4\left(s_4^*\right)$	$d_4\left(s_4^*\right)$
From	8	7	7	10
	9	5	5	10

In a similar way the following tables for the remaining stages are computed for cases when $r = 2$, $r = 3$, and $r = 4$ respectively.

Table for 3rd stage

From		To		$c_3(s_3^*)$	$d_3(s_3^*)$
		8	9		
	5	13	15	13	8
	6	12	9	9	9
	7	16	6	6	9

Table for 2nd stage

From		To			$c_2(s_2^*)$	$d_2(s_2^*)$
		5	6	7		
	2	17	15	-	15	6
	3	14	16	9	9	7
	4	-	11	14	11	6

The dash line means that there are no routes connecting cities 2 ad 4 of 2nd stage to cities 7 and 5 of 3rd stage respectively.

Table for 1st stage

From		To			$c_1(s_1^*)$	$d_1(s_1^*)$
		2	3	4		
	1	20	12	13	12	3

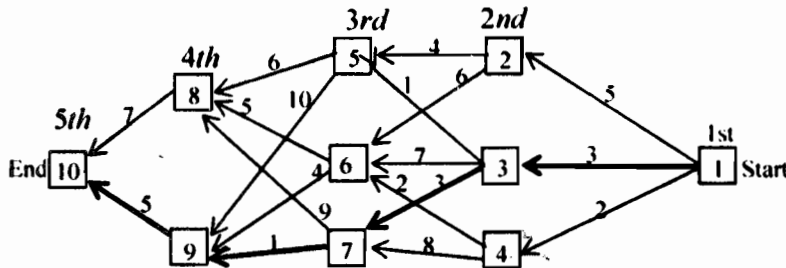**Step 3**

From the 1st stage table, $c_1(s_1^*) = c_1(1) = ₦1,200$ which is the minimum total cost. Recursively, the corresponding route is specified as follows:

Start $\boxed{1} \longrightarrow \boxed{3} \longrightarrow \boxed{7} \longrightarrow \boxed{9} \longrightarrow \boxed{10}$ End

Cost: 300   +300   +100   +500 = ₦1,200 = $c_1(s_1^*)$.

This minimum total cost route of ₦1,200 is indicated below in the network diagram.

4.   DISCUSSION

Equations (1) and (2) are two forms of $C_{n-r}(s_{n-r}^*)$ and while equation (1) expresses the principle of optimality, equation (2) expresses the enumeration and suboptimality considerations at each stage. By principle of optimality, we mean that an optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute the optimal policy with regard to the state resulting from the first decision. In other words, the decision made at each stage influences the next. In fact the decision made at each stage must take into account its effect not only on the next stage but also on the entire subsequent sequence of stages

The enumeration and suboptimal considerations at each stage has the advantage of less computational effort in the use of the algorithm when compared to other existing methods of solution. For example, in the numerical illustration there are 18 alternative paths (1x2x3x3x1) that can be taken. One can work out the total cost of each of these 18 routes to move from city 1 to city 10 and then the least cost route may be taken up. But this sort of complete

enumeration is not considered desirable. Another alternative approach would be to take the cheapest possible route between two successive cities: The cheapest route from city 1 is to city 4. then from city 4 to city 6. then from city 6 to city 9 and from city 9 to city 10. The total cost of this route would be N1300 But this route so worked out may not necessarily result in the overall optimal route because the choice of a least expensive route at one stage may force us to adopt a more expensive route at a later stage and thus may more than offset the earlier advantage This can happen when suboptimal cost over all cities in a stage is not taken into consideration at each stage.

5.   CONCLUSION

Dynamic programming is a mathematical technique that is applicable to many types of problems where series of interrelated decisions are required There is no single algorithm that can be used to solve all such problems i.e., a separate algorithm is needed for each type of problem. Hence we have developed and tested a dynamic programming algorithm for solving the stagecoach network problem. The algorithm refers to the cities in stages or subproblems and then solves the subproblems sequentially (usually working backward from the natural end of the problem) until the initial problem is finally solved. The steps of the algorithm are based on the principle of optimality. If the measure of effectiveness of the stagecoach network problem is that of maximum profit instead of minimum cost, a little modification of the algorithm will serve the purpose.

REFERENCES

Anderson, D.R., Sweeney, D. J. and Williams, T. A., 1997. An Introduction to Management Science, New Jersey: West Publishing Company.

Ebert, R. J. and Adam, E.E., 1978. Production and Operations Management: Concepts. Models and Behaviour, New Jersey: Prentice-Hall Inc.

Hillier, F.S. and Lieberman, G.J., 2001. An Introduction to Operations Research, San Francisco: Holden Day.

Marion, E.H., 1991. Project management from Idea to Implementation, London: Kogan Page Ltd.

Nunnally, S.W., 1980. Construction Methods and Management, New Jersey: Prentice-Hall Inc.

Okpamen, P.E., 2001. Production/Operations management" Benin City: Okosun Concept Ventures

Pilcher, R., 1992. Principles of Construction Management." London: McGraw-Hill International Ltd.

Render, B. and Stair, R.M., 1997. Quantitative Analysis for Management," New Jersey: Prentice Hall. Inc.

Taylor, B.W., 1996. Introduction to Management Science," New Jersey: Prentice-Hall.

Wagner, H. M., 1989. Principles of Operations Research," New Delhi: Prentice-Hall of India.

Weiss, H.J. and Gershon, M.E., 1993. Production and Operations Management," London. Allyn and Bacon Inc