GLOBAL JOURNAL OF MATHEMATICAL SCIENNCES VOL. 6, N0. 2, 2007: 105 - 109

COST CONSTRAINED MV-EFFICIENT OPTIMAL INCOMPLETE BLOCK DESIGNS

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(Received 17 October 2006; Revision Accepted 9 May 2007)

ABSTRACT

Cost constrained MV-efficient optimal incomplete block designs are highlighted. An algorithm on the deletion of treatments is presented, and implemented on computer using Matlab Software package, to determine the solution of a real life problem considered in the work.

KEYWORDS: Optimal design, Feasibility, C-matrix, Eigenvalue, MV-efficiency.

1.0 INTRODUCTION

In the paper of Stufken (1987), Morgan and Uddin (1991), Majumdar and Notz (1983), Majumdar and Hedayat (1985), Federov (1972), John and Mitchell (1977), the concept of Optimality is introduced in incomplete block design to construct A-, D-, and E-optimal incomplete block designs. In their work, experimental cost were not considered.

But Ogolime and Bamiduro (1998, 2000) introduced the cost of experiment in optimal incomplete block design to propose cost constrained A-, D-, and E-optimal incomplete block designs. However, Mbegbu and Etuk (2006) proposed generalized feasible solution of cost constrained optimal incomplete block designs and hence formulated a G-optimal BIBD with cost constraint.

In this paper, we shall propose cost constrained MV-efficient optimal incomplete block designs which has not been presented in the literature till date.

Now, let there exist a block design in v treatments and b blocks. Let r_i denote the replication of the ith treatment, k_j , the block size of the jth block and $N=(n_{ij})$ be the v×b incidence matrix of design, where n_{ij} denotes the number of times the ith treatment appears in the jth block, i=1,2,...,v; j=1,2,...,b. It is well known that under the usual homoscedastic fixed effects model, the normal equations for estimating the treatment effects are

$$Ct = Q \qquad 1.1)$$

where $C = diag(r_1,\cdots,r_v)$ -N $diag\left(k_1^{-1},\cdots,k_b^{-1}\right)N'$, t is the vector of treatment effects and Q is the vector of adjusted totals, given by $Q = T - N\,diag\left(k_1^{-1},\cdots,k_b^{-1}\right)B$, where T and B denote respectively the vector of treatment and block totals.

It can be shown that C is a singular matrix and thus, rank of C is at most v-1.

When rank of C is v-1, the design is called connected. A design is called equireplicate if $r_i = r$ for all i and binary if N has 0 and 1 as its elements. In this note, we shall consider only connected, equireplicate and binary balanced incomplete block design as an initial design.

The diagonal elements of the C-matrix are

$$c_{ii} = r_i - \sum_{j=1}^{b}\frac{n_{ij}^2}{k_j}, \quad i = 1,2,..,v \qquad (1.2)$$

and the off diagonal elements are

$$c_{ij} = -\sum_{\substack{h=1\\ i\neq j}}^{b}\frac{n_{ih}\,n_{jh}}{k_h}, \qquad (1.3)$$

i=1,2,...,v
j= 1,2,...,b

For binary, proper and equireplicate balanced incomplete block design

$$c_{ii} = r\left(1-\frac{1}{k}\right),\ i = 1,2,...,v \qquad (1.4)$$

and

$$c_{ij} = \frac{-\lambda}{k}, \qquad (1.5)$$

i=1,2,...,v
j= 1,2,...,b

Let $z_1(r), z_2(r), \ldots, z_v(r)$ be the eigenvalues of the C-matrix with $z_1(r) = 0$ and the nonnegative eigenvalues:

$$z_i(r) = \frac{rv(k-1)}{k(v-1)}, \quad i = 2,\ldots,v \qquad (1.6)$$

and $z_i^{-1}(r)$, which is the inverse of non-negative eigenvalues, and is obtained from equation (1.6). Hence

$$z_i^{-1}(r) = \frac{k(v-1)}{rv(k-1)}, \quad i = 2,3,\cdots,v \qquad (1.7)$$

More importantly, for every design, there exists $z_i(r)\ and\ z_i^{-1}(r)$. Clearly, $z_i^{-1}(r)$ are the eigenvalues of the inverse of C-matrix.

2.0 COST CONSTRAINED MV-EFFICIENT OPTIMAL INCOMPLETE BLOCK DESIGNS:

Assume the cost function to be linear (Ogolime and Bamiduro (1998) and let p_i be the set up cost with overhead cost for experimenting with r_i treatments. Let

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Φ be the total available resources for the experiment. Hence, the cost constraint,

$$\sum_{i=1}^{v} r_i p_i \leq \Phi \qquad (2.1)$$

The experiment must be carried out with the available resources.

By MV-Optimality, we mean

Minimize $\{Max\ z_i^{-1}(r)\}$ (2.2)

and it's efficiency defined as;

$$\text{MV-eff.} = \frac{\text{Min}\{\text{Max } z_i^{-1}(r)\}\text{ without Cost Constraint}}{\text{Min}\{\text{Max } z_i^{-1}(r)\}\text{ with Cost Constraint}} \qquad (2.3)$$

Hence, we have cost constrained MV-efficient Optimal Incomplete block design given as

Minimize $\{Max\ z_i^{-1}(r)\}$

subject to (2.4)

$$\sum_{i=1}^{v} r_i p_i \leq \Phi$$

$r_i > 0,\ \Phi > 0,\ p_i \geq 0$ are constant;

with it's efficiency as previously defined (see equation 2.3). The feasible solution of (2.4) is the set $\left\{ r_i^+ > 0 \middle/ \sum_{i=1}^{v} r_i^+ p_i \leq \Phi \right\}$ for all $i = 1,2,\ldots,v$ taken among designs with respect to Minimize $\{Max\ z_i^{-1}(r)\}$. To obtain a feasible solution to problem (2.4), firstly, identify a balanced incomplete block design that is near feasible, which is MV-optimal, otherwise, carry out the process of deletion of expensive treatments to meet available resources. The most expensive treatments will be deleted from the blocks to reduce the cost of experimentation to a manageable level. In the process of deleting treatments, the design may cease to be equireplicate or proper, and the balanced property may have been disturbed.

3.0 THE DELETION OF TREATMENTS ALGORITHM:

Given a balanced incomplete block design with parameters $\{v, b, k, r, \lambda\}$; the unit cost of each treatment i, p_i and Φ, the total resources for the experiment to be executed, then

Step 1: Choose a suitable balanced incomplete block design whose total cost of experimentation is near feasible. If it is feasible, that is $\sum_{i=1} r_i p_i \leq \Phi$, then stop. This is the design that satisfies cost constrained MV-optimal design. Otherwise proceed to step 2

Step 2: Delete m<k treatments (m=1,2,....,) from any block and test if the new set of treatment replications is feasible. Otherwise proceed to step 4.

Step 3: compute C_{n-m}, C_{n-m}^{-1}, $z_i^{-1}(r)$, and find the best combination of treatment replications that satisfy the cost constraint in terms of MV-optimality.

Step 4: set $m = m + 1$ and go to step 2. End.

4.0 PRACTICAL PROBLEM:

A farmer wishes to compare the effects of five types of fertilizer labelled A,B,C,D, and E on the yield of cassava. He wishes to run this experiment in a balanced incomplete block design with ten pieces of land serving as blocks. The cost of different types of fertilizer from the market survey is summarized in the table below:

	Types of Fertilizer				
	A	B	C	D	E
Cost per type of fertilizer (in thousand naira)	3	4	2	1	5

An agricultural agency undertakes to fund the experiment with the sum of ₦84,000 only. How can the farmer plan this experiment to meet on the restriction in funding and at the same time have an optimal result? It is assumed that the cost of labour, land, overhead cost is negligible.

4.1: Solution to the Practical Problem

The layout of the balanced incomplete block design is

A	A	A	A	A	A	B	B	B	C
B	B	B	C	C	D	C	C	D	D
C	D	E	D	E	E	D	E	E	E

$\lambda = 3$, $r = 6$, $k = 3$, $b = 10$, $v = 5$

The design matrix is given by

$$N_n' = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

The minimum cost of experimentation under the balanced incomplete block design is $\sum_{i=1} r_i p_i$ = ₦90,000 which is not feasible as this amount exceeds the budget (₦84,000) for the experiment. We now search for treatment that when deleted from the block will make this initial BIBD feasible in terms of satisfying the cost restriction. The cost analysis of deletion of one treatment is shown in the table 1 below.

Table 1: Cost Analysis of Deletion of one Treatment.

Treatment Deleted	Cost of Experiment $\sum_{i=1}^{5} r_i p_i$	Remark
A	5(3) + 6(4) + 6(2) + 6(1) +6(5) = ₦87,000	Not feasible
B	6(3) + 5(4) + 6(2) + 6(1) + 6(5) = ₦86,000	Not feasible
C	6(3) + 6(4) + 5(2) + 6(1) + 6(5) = ₦88,000	Not feasible
D	6(3) + 6(4) + 6(2) + 5(1) + 6(5) = ₦89,000	Not feasible
E	6(3) + 6(4) + 6(2) + 6(1) + 5(5) = ₦85,000	Not feasible

Hence, the minimum cost when a treatment is deleted is ₦85,000 , which is again not feasible.

We now embark on the deletion of two replicates of three most expensive treatments A, B, and E to satisfy cost restriction. We choose to delete either treatments B and E, A and B, or A and E. The cost of executing the experiment after the respective deletion is summarized in table 2 below.

Table 2: Cost Analysis of Deletion of Two Treatments

Treatments Deleted	Cost of Experiment: $\sum_{i=1}^{5} r_i p_i$	Remark
B and E	₦81,000	Feasible
A and B	₦83,000	Feasible
A and E	₦82,000	Feasible

With the feasibility condition being satisfied, by the algorithm we search for cost constrained MV-efficient optimal incomplete block design, and the results are shown in tables 3, 4 and 5 below, respectively.

Table 3: Deletion of Treatments B and E. Replicates: {6,5,6,6,5}

Blocks affected	MV-optimal Max $\{z_i^{-1}(r^+)\}$	MV-Efficiency	Remark
3,5	0.2609	76.66%	
3,6	0.2609	76.66%	
3,8	0.3000	66.67%	
3,9	0.3000	66.67%	
3,10	0.2789	71.71%	
1,3	0.2609	76.66%	
1,5	0.2727	73.34%	
8	0.2500	80.00%	MV-optimal
1,6	0.2609	76.66%	
1,8	0.2609	76.66%	
1,9	0.2727	73.34%	
1,10	0.2609	76.66%	
2,3	0.2609	76.66%	
9	0.2500	80.00%	MV-optimal
2,5	0.2609	76.66%	
2,6	0.2727	73.34%	
2,8	0.2727	73.34%	
2,9	0.2609	76.66%	
2,10	0.2609	76.66%	
7,8	0.2609	76.66%	
7,9	0.2609	76.66%	
3	0.2500	80.00%	MV-optimal
7,10	0.2727	73.34%	
8,9	0.3000	66.67%	
8,10	0.2609	76.66%	
9,10	0.2609	76.66%	

Table 4: Deletion of Treatments A and B; Replicates: {5,5,6,6,6}

Blocks affected	MV-optimal Max $\{z_i^{-1}(r^*)\}$	MV-Efficiency	Remark
1	0.2745	72.86%	
2	0.2500	80.00%	MV-Optimal
3	0.2500	80.00%	MV-Optimal
1,3	0.3000	66.67%	
1,2	0.3000	66.67%	
2,3	0.3000	66.67%	
1,4	0.2609	76.66%	
2,4	0.2609	76.66%	
3,4	0.2727	73.34%	
3,5	0.2609	76.66%	
5,7	0.2609	76.66%	
1,5	0.2609	76.66%	
1,6	0.2727	73.34%	
1,8	0.2609	76.66%	
1,9	0.2727	73.34%	
2,5	0.2727	73.34%	
2,6	0.2609	76.66%	
2,7	0.2609	76.66%	
2,8	0.2727	73.34%	
2,9	0.2609	76.66%	
3,6	0.2609	76.66%	
3,7	0.2727	73.34%	
3,8	0.2609	76.66%	
3,9	0.2609	76.66%	
4,7	0.2609	76.66%	
6,8	0.2609	76.66%	
6,9	0.2727	73.34%	
1,7	0.2609	76.66%	

Table 5: Deletion of Treatments A and E; Replicates: {5,6,6,6,5}

Blocks affected	MV-optimal Max $\{z_i^{-1}(r^*)\}$	MV-Efficiency	Remark
5	0.2500	80.00%	MV-Optimal
6,9	0.2609	76.66%	
6	0.2500	80.00%	MV Optimal
1,3	0.2609	76.66%	
6,10	0.2609	76.66%	
1,5	0.2609	76.66%	
1,6	0.2727	73.34%	
1,8	0.2727	73.34%	
1,9	0.2609	76.66%	
1,10	0.2609	76.66%	
2,3	0.2609	76.66%	
2,5	0.2727	73.34%	
2,6	0.2609	76.66%	
2,8	0.2609	76.66%	
2,9	0.2727	73.34%	
2,10	0.2609	76.66%	
3	0.2500	80.00%	MV-Optimal
3,5	0.3000	66.67%	
3,6	0.3000	66.67%	
3,8	0.2609	76.66%	
3,9	0.2609	76.66%	
3,10	0.2727	73.34%	
3,4	0.2727	73.34%	
4,5	0.2609	76.66%	
4,6	0.2609	76.66%	
4,8	0.2609	76.66%	
4,9	0.2609	76.66%	
4,10	0.2727	73.34%	
5,6	0.3000	66.67%	
5,8	0.2609	76.66%	
5,9	0.2727	73.34%	
5,10	0.2609	76.66%	
6,8	0.2727	73.34%	

4.2 Analysis of Deletion of Treatments

Deletion of treatments B and E from block 8, or block 9, or block 3 yields an MV-Optimal incomplete block design that is 80% efficient. Also deletion of treatments A and B from block 2, or block 3 yields an MV-Optimal incomplete block design that is 80% efficient. While deleting treatments A and E from block 5, or block 6, or block 3 yields an MV-Optimal incomplete block design that is 80% efficient.

The design matrices of two of the cost constrained MV-efficient optimal incomplete block designs identified in this problem are.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

5.0 CONCLUSION:

Any of the cost constrained MV-efficient Optimal incomplete block designs identified could serve as the best design that will give the Farmer an optimal result and at the same time meet the cost restriction. The Optimal designs are 80% efficient.

A typical cost constrained MV-efficient Optimal incomplete block design layout that could help the farmer to execute the experiment based on the available fund is shown below

Plots of Land (columns); Types of Fertilizer (rows)

A	A	A	A	A	A				
B	B	B				B		B	
C			C	C		C	C		C
	D		D		D	D		D	D
		E		E	E			E	E

This paper provides an experimenter an opportunity to plan before designing his experiment.

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