

A MODIFIED GENERALIZED GENERATING FUNCTION (GGF).

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ABSTRACT

A modified generalized generating function, $G_x(t)$, is herein proposed under an influence parameter ξ , satisfying $0 < \xi < 1$. This generating function is shown to define a functional relationship between the moment generating function $M_x(t)$ and the characteristic function $\Phi_x(t)$. Hence, it completely determines the distribution of a random variable. It is shown to possess an elastic radius of convergence, thus implying its existence, and it is uniformly continuous. It also has the capacity to generate moments as well as cumulants.

KEYWORDS: Modified Generalized Generating Function; influence parameter, elastic radius of convergence.

INTRODUCTION

Wilf (1994: 51) defines the generating function of a random variable as an expected value of a certain transformation of the variable. Feller (1968: 264) asserts that the theory of probability is heavily dependent on, and facilitated by the powerful methods of generating functions. Chung (1974: 128) identified three properties which are basic to all generating functions, and these are

- a) that the generating function determines the distribution of a random variable.
- b) that if X and Y are independent random variables, the generating function of $X + Y$ is equivalent to the product of their marginal generating functions
- c) that the moments of a random variable can be obtained from the derivatives of the generating function.

Stuart and Ord (1998: 80) adds that by appropriate power series expansion, moments of a random variable may be obtained as coefficients in the series expansion of the relevant generating function. We now proceed to introduce the generalized generating function (ggf)

Definition:

Let ξ be a real number satisfying $0 < \xi < 1$, and X a random variable (discrete or continuous).

$$\text{Let } G_x(t) = E \left[e^{(\xi + i(1-\xi))tX} \right] \quad (1)$$

where E is an operator representing mathematical expectation.

Then $G_x(t)$ is called the modified generalized generating function of the random variable X under the influence parameter ξ .

Properties of the modified generalized generating function, $(G_x(t))$

- a) Clearly, $G_x(t)$ defines a functional relationship between the moment generating function of the random variable X , $M_x(t)$ and its characteristic function, $\Phi_x(t)$. This is because

$$G_x(t) = M_x(\xi t) \cdot \Phi_x[(1-\xi)t] \quad (2)$$

and hence

$$G_x(t) = \begin{cases} M_x(t) & \text{if } \xi \rightarrow 1 \\ \Phi_x(t) & \text{if } \xi \rightarrow 0 \end{cases} \quad (3)$$

Since both $M_x(t)$ and $\Phi_x(t)$ determine the distribution of the random variable X (Stuart and Ord, 1998: 124), it follows that $G_x(t)$ also determines the distribution of X .

- b) From (2) $|G_x(t)| \leq |M_x(\xi t)| \cdot |\Phi_x[(1-\xi)t]| \leq |M_x(\xi t)|$ since $|\Phi_x| < 1$
 Now by power series expansion

$$M_x(\xi t) = 1 + \mu'_1 \xi t + \mu'_2 \frac{(\xi t)^2}{2!} + \mu'_3 \frac{(\xi t)^3}{3!} + \dots \quad (4)$$

which converges for $|t| < \frac{1}{\xi}$. But with an appropriate choice of ξ , the right hand side can be made as large as desired, hence the radius of convergence for $G_x(t)$ is elastic.

- c) If X and Y are two independent random variables, then observe that

$$\begin{aligned} G_{X+Y}(t) &= E \left\{ e^{(\xi + i(1-\xi))t(x+y)} \right\} \\ &= E \left[e^{(\xi + i(1-\xi))tX} e^{(\xi + i(1-\xi))tY} \right] \\ &= E \left[e^{(\xi + i(1-\xi))tX} \right] E \left[e^{(\xi + i(1-\xi))tY} \right] \\ &= G_X(t) \cdot G_Y(t) \quad (5) \end{aligned}$$

Hence $G_{X+Y}(t) = G_X(t) \cdot G_Y(t)$ whenever X and Y are independent.

- (d) On the continuity of $G_x(t)$ on R .
 Now consider $G_{x+n}(t) - G_x(t)$ as $n \rightarrow 0$

$$\begin{aligned}
 &= \left| e^{(\xi+(1-\xi)t)(x+h)} - e^{(\xi+(1-\xi)t)x} \right| \\
 &= \left| e^{(\xi h)} e^{i(1-\xi)t x} - e^{(\xi+(1-\xi)t)x} \right| \\
 &= \left| e^{(\xi+(1-\xi)t)x} - e^{(\xi+(1-\xi)t)h} - e^{(\xi+(1-\xi)t)x} \right| \\
 &\leq \left| e^{(\xi+(1-\xi)t)x} \left| e^{(\xi+(1-\xi)t)h} - 1 \right| \right| \\
 &\leq M \left| e^{(\xi+(1-\xi)t)h} - 1 \right| \text{ since } B_x(t) \text{ is bounded}
 \end{aligned}$$

The right hand side of the last expression is independent of X and can be made arbitrarily small for h sufficiently close to zero. Hence G_x(t) is uniformly continuous on R.

(e) On moments and cumulants generating capabilities of G_x(t).

If in (1) we put $\lambda = \xi+(1-\xi)$, we get $G_x(t) = E[e^{\lambda t x}]$ (6)

which by power series expansion becomes

$$G_x(t) = 1 + \mu'_1 \lambda t + \mu'_2 \frac{(\lambda t)^2}{2!} + \mu'_3 \frac{(\lambda t)^3}{3!} + \dots \dots (7)$$

Clearly G_x(0) = 1,

$$G'_x(0) = \lambda \mu'_1$$

$$G''_x(0) = \lambda^2 \mu'_2$$

And in general

$$G_x^{(k)}(0) = \lambda^k \mu'_k \dots (8)$$

As $\xi \rightarrow 1$ in (8), the appropriate raw moments are obtained.

In particular

$$E(X) = G'_x(0) \Big|_{\xi \rightarrow 1} \dots (9)$$

and $Var(x) = \left\{ G''_x(0) - [G'_x(0)]^2 \right\} \Big|_{\xi \rightarrow 1}$

Also from (7), it is obvious that G_x(t) generates moments with μ'_k as the coefficient of $\frac{(\lambda t)^k}{k!}$.

On cumulants generation, if we let

$$\begin{aligned}
 K_x(t) &= \ln G_x(t) \\
 &= \ln E[e^{\lambda t x}] \dots (10) \\
 &= \sum_{n=1}^{\infty} k_n \frac{(\lambda t)^n}{n!}
 \end{aligned}$$

(as in Stuart and Ord (1998: 86) and Weisstein (2005: 4)).

Hence, $\lambda^n k_n$ is the nth generalized cumulant of the random variable X.

Interestingly, using (3) on (10) we have

$$\ln G_x(t) = \begin{cases} \ln M_x(t) & \text{if } \xi \rightarrow 1 \\ \ln \phi_x(t) & \text{if } \xi \rightarrow 0 \end{cases} \dots (11)$$

Hence, the generalized cumulant generating function, K_x(t), is given by (10) and reduces to (11) as the influence parameter approaches its boundary values.

The modified generating function for some selected probability distributions.

Tabulated below are expressions for G_x(t) for some common probability distributions, along with their probability density functions, f(x). The most significant aspect of this is the existence of the ggf, for the Cauchy distribution though the computational process is quite tasking and requires contour integration [see Srinivasan and Mehata, 1981: 171].

Table 1. The modified generating function for some selected probability distributions

Distribution	f(x)	G _x (t)
Binomial	$\binom{n}{k} p^k q^{n-k}$	$\left(p e^{\xi t} e^{i(1-\xi)t} + q \right)^n$
Poisson	$\frac{\lambda^k e^{-\lambda}}{k!}$	$e^{\lambda} e^{\xi t} e^{i(1-\xi)t} - 1$
Geometric	$q^k p$	$P[1 - q e^{\xi t} e^{i(1-\xi)t}]^{-1}$
Exponential	$\theta e^{-\theta x}$	$\left[1 - \frac{\xi t}{\theta} - \frac{i(1-\xi)t}{\theta} \right]^{-1}$
Chi - square	$2^{-\frac{n}{2}} \Gamma\left(\frac{n}{2}\right) X^{\frac{n}{2}-1} e^{-\frac{1}{2}x}$	$[1 - 2\xi t - 2i(1-\xi)t]^{-\frac{n}{2}}$
Normal	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$	$e^{i\mu(\xi+(1-\xi)) + \frac{1}{2} \frac{(\xi+(1-\xi))^2}{\sigma^2} t^2}$
Cauchy	$\frac{\lambda}{\pi(\lambda^2 + x^2)}$	$e^{\frac{\lambda}{1-\xi} [\xi t + i(1-\xi)^2 t^2]}$

CONCLUSION

By defining a functional link between $M_x(t)$ and $\Phi_x(t)$, the generalized generating function provokes another line of thinking, and tends to usher in a simpler understanding of the inter-relationship between both. New challenges are also exhumed in that $G_x(t)$ may have wider implications for the theory of generating functions, and some related functions such as the Laplace and Fourier transforms.

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