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A SINGLE - ITEM REPLACEMENT DECISION MODEL FOR REPAIRABLE SPARE PARTS OVER AN INFINITE TIME HORIZON: AN ANALYTICAL SOLUTION

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ABSTRACT

In this paper, we present an analytical method for determining spare parts replacement over an infinite planning horizon. (The objective is to minimize the total system cost). We develop an exact and simple method for determining the time for equipment replacement or making decision about when to replace equipments, rather than that of continued maintenance of equipments We demonstrate that the optimum replacement time corresponds to when the cost of the next period is greater than the weighted average of expenditures already made. Numerical example is given to illustrate our proposed model

Keywords: Replacement, repairable infinite horizon spare parts, interest rate.

1.0 INTRODUCTION

In wide manufacturing environments product quality and yield is heavily influenced by equipment condition (Sloan 2004) Despite this fact, many researchers have either focussed on the maintenance or on the production, omitting the possibility of actually changing the machine state. As an item of equipment is kept in use for longer period of time it calls for increasing cost of repairs and maintenance. Hence a time is reached when it becomes more economical to replace the equipment altogether rather than maintaining it.

Literatures on the subject have shown that the problem of maintenance and replacement overlap in practice. Graves (1985) model was based on a multi-dimensional Markovian problem. His model gave exact results. However it is more difficult and computationally complex to solve Hau *et al.* (2007) developed a new approach for forecasting the intermittent demand of spare parts.

Lee (1987), Axsater, (1990), Kukreja *et al* (2001), Alfredsson and Verrijdt (1999) Yanagi and Sasaki (1992) and Hartanto *et al* (2005), Grahavac and Chakravarly (2001) gave heuristic methods in solving the replacement problem as a result of the complexity of the problem Sloan and Shanthikumar (2000) "developed a combined production and maintenance scheduling for a multi-product, single-machine production system Terje and Rommert (1997) presented a general framework for optimization of replacement times. Their models included various ages and block replacement models Lee (2005) developed a cost/benefit model for supporting investment strategies about inventory and preventive maintenance in an imperfect production system. Stanislaw (2005) presented a periodic review capacitated lot sizing model with limited backlogging and a possibility of emergency order He considered two re-supply modes; a regular mode and an emergency mode In relation with the system analyzed in this paper, two limitations are encountered in the previous models proposed by other authors, viz This treatment may be valid for the problems in which the repairable parts are installed on the machines standing on a fixed location, and or the repairable parts are installed on the machine that move from one location to another In this paper the limitations are removed

The primary focus of this paper therefore is to develop an interesting rule for taking optimal equipment replacement decisions where the equipments are on a fixed location, under the condition that the value of money is time dependent and the equipment is needed for infinite period in the future.

Our main contributions in this paper are; modeling demand, without delays and introducing interest rate. The paper is organised as follows: In the next section, we present the basic assumptions and notations used in the model. Section 3 describes the new decision modeling techniques and propositions. In section 4, we present an example of the model application. Section 5 gives the discussion and finally conclusion with directions for further studies in section 6.

2.0 NOTATIONS AND ASSUMPTIONS:

$f(n)$ – Total cost when an item of equipment is replaced after *nth* period.

r - Interest rate.

c_0 - Initial expenditure on the equipment

β - Discount factor

c_j - Cost for repairs/maintenance corresponding to the jth period

n^* - Optimal replacement time.

H - Infinite time horizon.

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Assumptions

- Charges for maintenance/repairs are paid at the beginning of the corresponding period.
- The equipment is needed over an infinite time horizon H.
- Money attracts a given interest rate r per period.
- Demand comes from an infinite source. This assumption is reasonable when the expected number of down machines is small relative to the total number of machines.
- There exits a series of equal periods numbered 1, 2, 3, . such that $c_2 > c_1$, $c_3 > c_2, \ldots c_{n+1} > c_n$
- The period n is selected in such a way that it is finite

3.0 MODEL FORMULATION

We first note that with interest rate r, the present value (PV) of an amount c to be paid m periods in the future is,

$$pv = \frac{c}{(1+m)^m}$$

With the initial expenditure c_0 and interest rate r, we modeled the relationship that the total cost incurred when the equipment is replaced after n period is:

$$\begin{aligned} f(n) &= \left[c_0 + c_1 + \frac{c_2}{(1+r)} + \frac{c_3}{(1+r)} + \cdots + \frac{c_n}{(1+r)^{n-1}}\right] \\ &+ \left[\frac{c_0}{(1+r)^n} + \frac{c_1}{(1+r)^n} + \frac{c_2}{(1+r)^{n+1}} + \cdots + \frac{c_n}{(1+r)^{2n-1}}\right] \\ &+ \left[\frac{c_0}{(1+r)^{2n}} + \frac{c_1}{(1+r)^{2n}} + \frac{c_2}{(1+r)^{2n+1}} + \cdots + \frac{c_n}{(1+r)^{3n}}\right] + \cdots \end{aligned} \tag{1}$$

This can be simplified as

$$f(n) = \left[c_0 + \sum_{i=1}^{n} \frac{c_i}{(1+r)^{i-1}}\right] + \frac{1}{(1+r)^n}\left[c_0 + \sum_{i=1}^{n} \frac{c_i}{(1+r)^{i-1}}\right] \frac{1}{(1+r)^{2n}}\left[c_0 + \sum_{i=1}^{n} \frac{c_i}{(1+r)^{i-1}}\right] + \cdots$$ Thus,

$$f(n) = \left[c_0 + \sum_{i=1}^{n} \frac{c_i}{(1+r)^{i-1}}\right]\left[1 + \frac{1}{(1+r)^n} + \frac{1}{(1+r)^{2n}} + \cdots\right]$$

But $\left[1 + \left(\frac{1}{1+r}\right)^n + \left(\frac{1}{1+r}\right)^{2n} + \cdots\right]$ is a geometric series summing to infinity

Recall that the sum to infinity and r is the common ratio given by

$$1 + \frac{1}{(1+r)^n} + \frac{1}{(1+r)^{2n}} + \cdots = \frac{1\left(\frac{1}{(1+r)^n}\right)^n}{1 - \frac{1}{(1+r)^n}} = \frac{1 \frac{1}{(1+r)^{n^2}}}{1 - \frac{1}{(1+r)^n}}$$

As n becomes large, then $\frac{1}{(1+r)^n} \to 0$

$$f(n) = \frac{\left[c_0 + \sum_{i=1}^{n} \frac{c_i}{(1+r)^{i-1}}\right]}{1 - \frac{1}{(1+r)^n}}, \text{ provided } \frac{1}{1+r} < 1 \tag{2}$$

Let $\beta = \frac{1}{1+r}$, equation (2) reduces to

$$f(n) = \frac{c_0 + \sum_{i=1}^{n} \beta^{i-1} c_i}{1 - \beta^n} \tag{3}$$

The period n is selected in such a way that it is finite.

Equation (3) represents the amount that should be available at the beginning of these periods to replace the equipment every n period (say n years)

The minimum of $f(n)$ which corresponds to the optimum replacement period can be found by tabulating the values of $f(n)$ for various values of n

However, we shall demonstrate a rule for obtaining the minimum of $f(n)$ in the next proposition

Proposition

Suppose $c_2 > c_1,\ c_3 > c_2,\ c_4 > c_3, \cdots, c_{n+1} > c_n$, then, the value of n for which the inequality,

$$c_{n+1} > \frac{c_0 + c_1 + c_2\beta + \cdots + c_n\beta^{n-1}}{1+\beta+\beta^2+\cdots+\beta^{n-1}},$$

first holds, corresponds to the optimum replacement time.

The **proposition above leads to the rule: Replace** the equipment when the cost of the *next period* is greater than the **weighted mean of all expenditure already made.**

Proof:

The proof of our proposition shall be in two cases

Suppose the minimum of $f(n)$ exists for value of n^*, then we must have

Case I: $f(n^*+1) - f(n^*) > 0$

Case II: $f(n^*-1) - f(n^*) > 0$

Case I: $f(n^*+1) - f(n^*) > 0$

If we replace n by $(n+1)$ in equation (3), we have

$$f(n+1) = \frac{c_0 + \sum_{i=1}^{n+1}\beta^{i-1}c_i}{1-\beta^{n+1}}$$

$$= \frac{c_0 + \sum_{i=1}^{n}\beta^{i-1}c_i + \beta^n c_{n+1}}{1-\beta^{n+1}} \tag{4}$$

But from (3),

$$c_0 + \sum_{i=1}^{n}\beta^{i-1}c_i = (1-\beta^n)f(n) \tag{5}$$

Substitute (5) in (4)

$$f(n+1) = \frac{(1-\beta^n)}{1-\beta^{n+1}}f(n) + \frac{\beta^n c_{n+1}}{1-\beta^{n+1}} \tag{6}$$

Thus

$$f(n+1) - f(n) = \frac{(1-\beta^n)}{1-\beta^{n+1}}f(n) + \frac{\beta^n c_{n+1}}{1-\beta^{n+1}} - f(n)$$

$$= \left[\frac{1-\beta^n - 1 + \beta^{n+1}}{1-\beta^{n+1}}\right]f(n) + \frac{\beta^n c_{n+1}}{1-\beta^{n+1}}$$

From which we have

$$f(n+1) - f(n) = \frac{\left[\beta^{n+1} - \beta^n\right]f(n) + \beta^n c_{n+1}}{1-\beta^{n+1}} \tag{7}$$

If case 1 holds, then,

$$\frac{\left[\beta^{n^*+1} - \beta^{n^*}\right]f(n^*) + \beta^{n^*} c_{n^*+1}}{1-\beta^{n^*+1}} > 0 \tag{8}$$

where $1-\beta^{*n+1} > 0$.

Divide (8) by β^{n^*} we have

$$\frac{(\beta-1)f(n^*)+c_{n^*+1}}{1-\beta^{n^*+1}} > 0$$

Since $\beta = \frac{1}{1+r}$, and for $r>0$, then $\beta^{n^*+1} = \left(\frac{1}{1+r}\right)^{n^*+1} < 1$

We can write,

$$f(n^*)(\beta-1) + c_{n^*+1} > 0$$
$$c_{n^*+1} > -[-(1-\beta)f(n^*)]$$

which follows that:

$$\frac{c_{n^*+1}}{1-\beta} > f(n^*) \tag{9}$$

Thus

$f(n^*+1)-f(n^*) > 0$ is equivalent to $\frac{c_{n^*+1}}{1-\beta} > f(n^*)$

Case II when $f(n^*-1)-f(n^*) > 0$

Note $f(n) = \frac{c_0 + \sum_{j=1}^{n}\beta^{j-1}c_j}{1-\beta^n} \Rightarrow (1-\beta^n)f(n) = c_0 + \sum_{j=1}^{n}\beta^{j-1}c_j$

Replace n by $(n-1)$ in equation (3), we have

$$f(n-1) = \frac{c_0 + \sum_{j=1}^{n-1}\beta^{j-1}c_j}{1-\beta^{n-1}} \Rightarrow (1-\beta^{n-1})f(n-1) = c_0 + \sum_{j=1}^{n-1}\beta^{j-1}c_j \tag{10}$$

$$f(n-1) = \frac{c_0 + \sum_{j=1}^{n}\beta^{j-1}c_j - \beta^{n-1}c_n}{1-\beta^{n-1}} = \frac{c_0 + \sum_{j=1}^{n}\beta^{j-1}c_j}{1-\beta^{n-1}} - \frac{\beta^{n-1}c_n}{1-\beta^{n-1}}$$
$$= \frac{1-\beta^n}{1-\beta^{n-1}} - \frac{\beta^{n-1}c_n}{1-\beta^{n-1}}$$

Thus,

$$f(n-1)-f(n) = \frac{(1-\beta^n)}{(1-\beta^{n-1})}f(n) - f(n) - \frac{\beta^{n-1}c_n}{1-\beta^{n-1}} \tag{11}$$

Since $\beta = \frac{1}{1+r}$, then $0<\beta<1$, for $r>0$

From (10),

$$c_0 + \sum_{j=1}^{n}\beta^{j-1}c_j = (1-\beta^{n-1})f(n-1) + \beta^{n-1}c_n \tag{12}$$

$$= \frac{(1-\beta^n-1+\beta^{n-1})f(n)}{(1-\beta^{n-1})} - \frac{\beta^{n-1}c_n}{1-\beta^{n-1}} \tag{13}$$

If the inequality in case II holds, then,

$$= \frac{(\beta^{n-1}-\beta^n)f(n)}{(1-\beta^{n-1})} - \frac{\beta^{n-1}c_n}{1-\beta^{n-1}} > 0 \tag{14}$$

Dividing through (14) by β^{n-1}, we have

$$\frac{(1-\beta)f(n)-c_n}{1-\beta^{n-1}} > 0 \tag{15}$$

Since, $1-\beta^n > 0$, (15) becomes

$$(1-\beta)f(n-1)-c_{n^*} > 0$$

This is equivalent to,

$$-c_{n^*} > -(1-\beta)f(n^*-1)$$

which implies that

$$\frac{c_{n^*}}{1-\beta} < f(n^*-1) \qquad (16)$$

Thus

$$f(n^*-1)-f(n^*) > 0 \text{ is equivalent to}$$

$$\frac{c_{n^*}}{1-\beta} < f(n^*-1)$$

Now, going back to the expression for $f(n)$ in equation (9), we obtain.

$$c_{n^*+1} > \frac{(1-\beta)(c_0+c_1+\beta c_2+\beta^2 c_3+\cdots+\beta^{n^*-1}c_{n^*})}{1-\beta^{n^*}} \qquad (17)$$

But $$\frac{(1-\beta^n)}{1-\beta} = 1+\beta+\beta^2+\cdots+\beta^{n-1}$$

And so equation (12) becomes

$$c_{n^*+1} > \frac{c_0+c_1+\beta c_2+\beta^2 c_3+\cdots+\beta^{n^*-1}c_{n^*}}{1+\beta+\beta^2+\cdots+\beta^{n^*-1}} \qquad (18)$$

In a similar manner, substituting the expression for $f(n)$ in equation (16) and replacing n with $n-1$, we have,

$$c_{n^*} < \frac{(1-\beta)}{(1-\beta^{n^*-1})}\left(c_0+c_1+\beta c_2+\beta^2 c_3+\cdots+\beta^{n^*-2}c_{n^*-1}\right) \qquad (19)$$

But, $$\frac{(1-\beta^{n-1})}{1-\beta} = 1+\beta+\beta^2+\cdots+\beta^{n-2}$$

So equation (19) becomes

$$c_{n^*} < \frac{c_0+c_1+\beta c_2+\beta^2 c_3+\cdots+\beta^{n^*-2}c_{n^*-1}}{1+\beta+\beta^2+\cdots+\beta^{n^*-2}} \qquad (20)$$

The RHS of equations (18) and (20) are actually the weighted means of all the costs from period $j=1(1)n$ for (18) and from $j=1(1)(n-1)$ for (20). The weights are the discount factors $1, \beta, \beta^2, \cdots$ corresponding to each period. Hence we arrive at the following rule: Replace the equipment when

$$c_{n^*+1} > \frac{c_0+c_1+\beta c_2+\cdots+\beta^{n^*-1}}{1+\beta+\beta^2+\cdots+\beta^{n^*-1}}$$

where the value of n for which the above inequality holds is the optimum replacement time This is now our new equation for optimal replacement time for equipments as at when due.

4.0 NUMERICAL ILLUSTRATION

In this section, we illustrate the use of the model by a practical example (Arnold and Kaufmann (1963). Let the interest rate over time be $r=6\%$, and other parameters are given in the Table 1 below. Table 2 shows the numerical computations.

Table 1: Parameters for the problem with interest rate 6%.

Year	0	1	2	3	4	5	6	7	8	9	10	11	12
c_j (₦) Thousand	1,000,000	50	60	70	90	120	150	180	210	240	300	400	500

The problem is: When do we replace the existing item?

Table 2: The Numerical computations and solution.

Year	c_j thousand	β^{j-1}	$c_j\beta^{j-1}$	$c_0+\sum_{j=1}^{n} c_j\beta^{j-1}$	$\sum_{j=1}^{n}\beta^{j-1}$	$f(n)$	$\dfrac{c_0+\sum_{j=1}^{n} c_j\beta^{j-1}}{\sum_{j=1}^{n}\beta^{j-1}}$
1	50	1.000	50.0	1,050	1.000	18,421	1,050
2	60	0.943	56.6	1,107	1.943	10,064	570.0
3	70	0.890	62.3	1,169	2.833	7,306	413.0
4	90	0.840	75.6	1,245	3.673	5,986	339.0
5	120	0.792	95.0	1,340	4.465	5,296	300.0
6	150	0.747	112.1	1,452	5.212	4,922	279.0
7	180	0.705	126.9	1,579	5.917	4,713	267.0
8	210	0.665	139.7	1,718	6.582	4, 606	261.0
9*	240*	0.627*	150.5*	1,869*	7.209*	4,581*	259.0*
10	300	0.592	177.6	2,046	7.801	4,629	262.0
11	400	0.558	223.2	2,270	8.359	4,799	272.0
12	500	0.526	263.0	2,533	8.886	5,026	285.0

For clarity purpose of how the values in Table 2 were obtained, we shall be inquisitive to quickly show how in the first year the values were gotten.

Year 1: [Note $c_0 = 1000 \ \textit{in thousand}$]

$$c_j = c_1 = 50$$

$$\beta^{j-1} = \beta^{1-1} = \beta^0 = 0.943^0 = 1$$

$$c_j\beta^{j-1} = c_1\beta^0 = 50\times 1 = 50.0$$

$$c_0 + \sum_{j=1}^{n} c_j\beta^{j-1} = c_1 + \sum_{j=1}^{1} c_j\beta^{j-1} = 1000 + 50 = 1.050$$

$$\sum_{j=1}^{n}\beta^{j-1} = \sum_{j=1}^{1}\beta^{j-1} = \beta^0 = 1.000$$

$$f(n) = \frac{c_0 + \sum_{j=1}^{n} c_j\beta^{j-1}}{1-\beta^n} = \frac{1,050}{1-\beta^1} = \frac{1,050}{1-(0.943)^1} = \frac{1,050}{1-0.943}$$

$$f(n) = 18,421 = f(1)$$

$$\frac{c_0 + \sum_{j=1}^{n} c_j\beta^{j-1}}{\sum_{j=1}^{n}\beta^{j-1}} = \frac{1,050}{1.00} = 1,050$$

Similarly this applies to other columns in the table II.

5.0 REMARK

From Table II, the minimum of $f(n)$ (in column 7) occurs at $n^* = 9$ and the corresponding value of $f(n)$ is ₦ 4,581. Observe that from column 8 of Table II, the minimum of $f(n)$ is achieved when the cost of the next period, $((n^* + 1 = 10))$ given as c_{10} is greater than the weighted mean of all expenditures already made That is when

$$c_{10} > \frac{c_0 + \sum_{j=1}^{9} c_j\beta^{j-1}}{\sum_{j=1}^{9}\beta^{j-1}} \qquad \text{that is, } 300 > 259.$$

Hence, since the cost of the 10th year c_{10} is greater than the weighted mean of all expenditures made up to the 9th year, then the optimum replacement time is the 9th year. It follows that computing the values of $f(n)$ for various values of n coincides with the use of our rule.

Note that the cost of the 11th year is also greater than the weighted mean of all expenditures made up to the 10th year, i.e ($c_{11} = 400 > 262$). A similar situation also holds for the 12th year. However, consideration is given to when the situation first occurs. Thus, since for the first time, the cost of the 10th is greater than the weighted mean of all expenditures made up to the 9th year, then we take the 9th year as the optimum replacement time That is $n^* = 9$

6.0 CONCLUSION

We have developed an analytical solution methodology for determining the replacement time for repairable spare parts of a single-item over an infinite time horizon *H*. We demonstrated that the proposed model is simple rule for replacement decision. Obviously, the solution from the computation of $f(n)$ for all possible values of *n*, which minimizes the total cost, $f(n)$ is taken as the optimum replacement time.

Hence if all the assumptions we have considered hold true, replacement of a defender (existing equipment) is due when the cost of keeping it for the next period is greater than the weighted mean of all expenditure already made

Moreover, our method is computationally efficient and can be used to analyze variety of problems. Further research is needed for repairable parts on machines that move from place to place and the cost implication suppose the parts are not readily available. We have presented all mathematical demonstrations in such a way that this work can be understood by other researchers in other fields of study

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