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ERROR AND PROCESS ESTIMATION OF ARCH (1) MODEL CORRUPTED BY AR(1) PROCESS

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ABSTRACT

We showed how autocovariance functions can be used to estimate the ARCH(1) process corrupted by AR(I) errors. we performed simulation studies to demonstrate our findings. The studies showed that our model was able to very closely estimate the required ARCH process in the presence of AR(1) errors.

KEY WORDS: AR, ARMA, ARCH, Error and Process Estimation

1.0 ARCH FRAME WORK.

Let $\{y_t\}$ denote a stochastic process with mean μ_t, then the error term is defined as (see Bollerslev, Engle and Nelson (1994))

$$\varepsilon_t = y_t - u_t$$

Under the assumption of constant variance, and correct model specification,
ε_t will be distributed as Z_t where Z_t is any symmetric distribution. However, under a time- varying variance condition, ε_t will be expressed as a product, ie

$$\varepsilon_t = Z_t\ h_t^{1/2}$$

where h_t is the conditional variance at time t and Z_t is any symmetric distribution. Bollerslev, Engle and Nelson (1994) defines the ε_t process to follow an Autoregressive Conditional Heteroscedascity (ARCH) model ARCH process if

$$E_{t-1}(\varepsilon_t)=0 \quad t=1,2,$$

In addition, the conditional variance is

$$h_t = \operatorname{var}_{t-1}\{\varepsilon_t\} = \mathrm{E}_{t-1}\{\varepsilon_t^{\,2}\}.$$

where $\mathrm{E}_{t-1}(.)$ denotes the conditional expectation when the conditioning set is compose of information up to time t-1

Engle's (1982) ARCH(q) model is presented as ARCH model as a linear function of the past squared disturbances. That is

$$\varepsilon_t^2 = z_t^2 h_t,$$

and

$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon^2_{t-i}$$

2.0 PROBLEM FORMULATION

Consider the ARCH (1,1) model equation

$$h_t = \alpha_0 + \alpha\varepsilon_{t-1}^2, \tag{1}$$

and

$$\varepsilon_t^2 = Z_t^2 h_t,$$

with parameter constraints

$$\alpha_0 > 0, \alpha \geq 0$$

These constraints are meant to ensure that the variance is positive.
Equation (1) admits transformation to AR (1) model through the substitution

$$a_t = \varepsilon_t^2 - h_t,$$

to get

$$\varepsilon_t^2 = \alpha_0 + \alpha\varepsilon_{t-1}^2 + a_t, \tag{2}$$

or

$$(1-\alpha L)\varepsilon_t^2 = \alpha_0 + a_t \tag{3}$$

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We define

ε_t^2 as an unobservable process of interest,

a_t as a white noise process i.e a_t is distributed as $(0, \sigma_a^2)$,

L as a backward shift operator i.e $La_t = a_{t-1}$ and

α is a weight parameters.

Eni and Etuk (2006a,b) have used autocovariance functions to show that the transformation of equation (1) to equation (3) through the substitution $a_t = \varepsilon_t^2 - h_t$ is justified.

Our interest is now in the case where although ε_t^2 is unobservable, we can estimate it through

$$\varepsilon_t^2 = g^2{}_t - b_t \,, \tag{4}$$

where

$g^2{}_t$ is an observed process,

b_t is an error component following AR(1) process.

Substituting equation (4) into (3), we have

$$(1-\alpha L)g_t = \alpha_0 + a_t + (1-\alpha L)b_t \,. \tag{5}$$

Since b_t is modeled as AR (1)

$$(1-\phi L)b_t = e_t$$

or

$$b_t = \frac{e_t}{(1-\phi L)}, \tag{6}$$

where

e_t is white noise process independent of a_t

Substituting equation (6) into (5),we have

$$(1-\alpha L)(1-\phi L)g^2{}_t = A_0 + ((1-\phi L)a_t + (1-\alpha L)e_t \,,$$

where

$$A_0 = \alpha_0(1-\phi),$$

$$g^2{}_t = A_0 + (\phi+\alpha)g^2{}_{t-1} - \phi\alpha g^2{}_{t-2} + a_t - \phi a_{t-1} + e_t - \alpha e_{t-1} \,. \tag{7}$$

Our main objective is to estimate the process ε_t^2 and error b_t through g_t^2.

Moran (1971) has shown that if the ratio $\lambda = \frac{\sigma_a^2}{\sigma_a^2}$ is known, then the maximum likelihood estimates for the parameter set can be found. The maximum likelihood estimates for the case where both σ_a^2 *and* σ_b^2 are known (the so called "over verification case") are estimated by Barnett (1967) by directly solving the likelihood equation. Chan and Mak (1979) obtained the maximum likelihood estimates for the case where both σ_a^2 *and* σ_e^2 are unknown and where the observations are replicated.

Our interest is to use autocovariance function to estimate the parameter values of the real series even where the ratio $\lambda = \frac{\sigma_a^2}{\sigma_e^2}$ is unknown Eni, et al (2007a) have used the same method to isolate errors of of AR(1) corrupted with MA(1) process. Also Eni, et al (2007b) have considered the case of IMA(1) with white noise. In a similar case, Eni (2006) have considered the case of GARCH(1,1) model with white noise errors using the proposed method.

Taking expectation of g_t in equation (7), we have

$$E(g^2{}_t) = A_0 + (\phi+\alpha)E(g^2{}_t) - \phi\alpha E(g^2{}_t),$$

or

$$E(g^2{}_t) = \frac{A_0}{(1-\phi)(1-\alpha)}, \tag{8}$$

where

$$E(g^2{}_{t-i}) = E(g_t^2) \qquad i = 1,2...,$$

$$E(a_t) = E(a_{t-1}) = E(e_t) = E(e_{t-i}) = 0,$$

see Hamilton (1994).

Multiplying (7) by $g^2{}_t$ and taking expectation, we have

$$v_0 = \frac{A_0^2}{(1-\phi)(1-\alpha)} + (\phi+\alpha)v_1 - \phi\alpha v_2 + \sigma_a^2 - \phi E(g^2{}_t a_{t-1}) + \sigma_e^2 - \alpha E(g^2{}_t e_{t-1}), \qquad (9)$$

where, by definition

$$E(e_t e_{t-i}) \quad or \quad E(a_t a_{t-1}) = \begin{cases} 0 & if \quad i \neq 0 \\ \sigma_e^2 \ or \ \sigma_a^2 & respectively \ if \ i = 0 \end{cases} \qquad (10)$$

$E(g^2{}_t g^2{}_{t-i}) = v_i$, see Box and Jenkins (1976). (11)

$E(g^2{}_{t-i} a_{t-i}) \ or \ E(g^2{}_{t-i} e_{t-i}) = \sigma_a^2 \ or \ \sigma_e^2$, respectively. (12)

Multiplying equation (7) by a_{t-1}, e_{t-1} and taking expectation using (10), (11) and (12) we have

$$E(g^2{}_t a_{t-1}) = \alpha\sigma_a^2, \qquad (13)$$

$$E(g_t^2 e_{t-1}) = \phi\sigma_a^2. \qquad (14)$$

Substituting equations (13), (14), and into (9)
we have

$$\frac{A_1}{(1-\alpha)(1-\phi)(1-\alpha\phi)} = \sigma_a^2 + \sigma_e^2, \qquad (15)$$

where

$$A_1 = (1-\alpha)(1-\phi)\{ v_0 - (\phi+\alpha)v_1 + \phi\alpha v_2 \} + A_0^2$$

Multiplying equation (7) by $g^2{}_{t-1}$ and taking expectations using equations (10), (11), (12) and (13), we have

$$\frac{A_2}{\{1-\alpha-\phi+\alpha\phi\}} = \alpha\sigma_e^2 - \phi\sigma_a^2, \qquad (16)$$

where

$$A_2 = (1-\phi)(1-\alpha)\{ v_1 - (\phi+\alpha)v_0 - \alpha\phi v_1 \} + A_0^2. \qquad (17)$$

Solving simultaneously equations (15) and (16) for σ_e^2 and σ_a^2, we have

$$\sigma_e^2 = \frac{A_1\phi(1-\alpha-\phi+\alpha\phi) + A_2(1-\alpha)(1-\phi)(1-\alpha\phi)}{(1-\alpha)(1-\phi)(1-\alpha\phi)(1-\alpha-\phi+\alpha\phi)(\phi-\alpha)}, \qquad (18)$$

$$\sigma_a^2 = \frac{A_1\alpha(1-\alpha-\phi+\alpha\phi) - A_2(1-\alpha)(1-\phi)(1-\alpha\phi)}{(1-\alpha)(1-\phi)(1-\alpha\phi)(1-\alpha-\phi+\alpha\phi)(\alpha-\phi)}. \qquad (19)$$

3.0 PROCESS ESTIMATION

The model (7) is theoretical since the traditional ARMA model does not make provision of two set of white noise errors as we have in equation (7).In practice, we will observe (7) as the ARMA (2,1) model .

$$g^2{}_t = C + \Omega_1 g^2{}_{t-1} - \Omega_2 g^2{}_{t-2} + U_t - \theta_1 U_{t-1} \qquad (20)$$

where

U_t is a white noise process

C is a constant

$\Omega_1, \Omega_2 \ and \ \theta_1$, are weight parameters.

We can obtain good estimates of the parameters $C, \Omega_1, \Omega_2 \ and \ \theta_1$ found in equation (20) through the maximum likelihood method. (See Box and Jenkins (1976) for example).

However, our interest is to estimate the parameters $\alpha \ and \ \phi$ in equation (7). To do this, we note that by comparing equations (7) and (20)

$C = A_0$ (i)

$$\Omega_1 = \alpha + \phi \qquad (ii)$$
$$\Omega_2 = \alpha\phi \qquad (iii)$$

while the white noises are equated as

$$U_t - \theta_1 U_{t-1} = a_t + e_t - (\phi\, a_{t-1} + \alpha\, e_{t-1}) \quad (iv)$$

also from (6b), we have

$$A_0 = \alpha_0(1-\phi) \Rightarrow \alpha_0 = \frac{C}{1-\phi}$$

Substitution of $\alpha = \Omega_1 - \phi$ into (iii) will result into the quadratic equation

$$\phi^2 - \phi\Omega_1 + \Omega_2 = 0 \qquad (v)$$

The results of (v) will be substituted into (ii) to obtain the values of α. This will give two pairs of ϕ, α results. However, we consider the fact that equations (18) and (19) must be positive and recommend the choice of the ϕ, α pair that will make both equations positive.

We follow Box and Jenkins (1976) to compute the variance and autocovariances, v_0, v_1 and v_2 from the observe data g_t^2 using the formula

$$v_i = \frac{1}{N}\sum_{t=1}^{N-i}(g_t^2 - \mu)(g_{t+i}^2 - \mu), \qquad (21)$$

where

$i = 0,1,2$

$$\mu = \frac{1}{N}\sum_{t=1}^{N} g_t^2$$

N is the total number of data points

With the parameters, α and ϕ as well as the autocovariances, v_0, v_1 and v_2 known, we can use equations (18) and (19) to estimate σ_a^2 and σ_e^2. Hence we can generate normal random processes with mean zero and variance σ_a^2 and with mean zero and variance σ_e^2 to represent the white noise processes a_t and e_t respectively. We can do this by using the random number generator of any software package like MATLAB, for example

With our knowledge of α_0 (see i), α and the white noise process a_t, we can now estimate the process of interest ε_t^2 using equation (2) which results into the recursion below.

$$\begin{aligned}
\varepsilon_1^2 &= \alpha_0 + a_1\\
\varepsilon_2^2 &= \alpha_0 + a_2 + \alpha(\alpha_0 + a_1)\\
\varepsilon_3^2 &= \alpha_0 + a_3 + \alpha(\alpha_0 + a_2) + \alpha^2(\alpha_0 + a_1)\\
&\vdots\\
\varepsilon_t^2 &= \sum_{i=0}^{t-1}\alpha^i(\alpha_0 + a_{t-i}), \quad t = 1,2,\ldots
\end{aligned} \qquad (22)$$

In addition, we can also estimate the error process using equation (6). This will result to the

$$\begin{aligned}
b_1 &= e_1\\
b_2 &= e_2 + \phi e_1\\
&\vdots\\
b_t &= e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots + \phi^{t-1} e_1\\
&= \sum_{i=0}^{t-1}\phi^i e_{t-i}, \quad t = 1,2,\ldots
\end{aligned} \qquad (23)$$

4.0 ILLUSTRATION

We use the NORMRND facility in MATLAB5(1999) to simulate 1400 data points each of a_t and e_t. a_t is generated with mean 0 and variance 3.30 while e_t is generated with mean 0 and variance 1 15. To make a_t and e_t white noise, the means must be zeros while the variances can be any suitable positive values. From equations (2) and (6), we ensure stationarity by choosing α =0.75, $\phi = .84$ and $\alpha_0 = 0.12$.(See Box and Jenkins (1976) for example).

We use this to simulate 1,400 data points each of the following.

(1) ε_t^2 (Assumed unobserved) $= 0.10 + 0.73\varepsilon_{t-1}^2 + a_t$, t=1, 2 . 1400. (see equations (2))

The values of the process ε_t^2 was obtained using recursion (22)

(2) The AR(1) errors $b_t = 0.83b_{t-1} + e_t$, t=1,2,...1400. (see equations (6))

The values of the error process b_t was obtained using recursion (23)

(3) The observed value g_t^2 is the sum of (1) and (2)

We discarded the first 200 data points to avoid initialization problems. This leaves us with 1200 data points for our analyses. However, due to space limitations, only ten data points from t=201 to t=210 are shown in Table1 for ε_t^2 and g_t^2 as simulated processes.

Our objective is to estimate the process ε_t^2 (Assumed unobserved or unknown) through the observed process g_t^2. We also use g_t^2 to estimate the error process b_t.

We compute the first three autocovariances of the observe process g_t^2 using formula (21). We obtain the result below

$$v_0 = 2.1674,\quad v_1 = 0.9768,\quad v_2 = 0.4612 \qquad (24)$$

The Mcleod and Sales (1983) maximum likelihood estimates facilities in STATISTICA (1995) was used to get the following parameter values(found in equation (20)).

$$\Omega = 1.59,\quad \Omega_1 = 0.63,\quad \theta_1 = 1.43,\quad C = .03 \qquad (25)$$

From (i),(ii),(iii)and (iv) in section 3.0, we obtain the following estimates

$$\alpha_0 = 0.12,\quad \alpha = 0.75,\quad \phi = .84 \qquad (26)$$

We substitute α =0.75 and $\phi = .84$ obtained in (26) and v_0, v_1 and v_2 obtained in (24) into equations (18) and (19) to estimate the variances $\hat{\sigma}_e^2$ and $\hat{\sigma}_a^2$ of the white noise processes e_t and a_t respectively. We obtained the results as

$$\hat{\sigma}_e^2 = 1.11,\quad \hat{\sigma}_a^2 = 3.33 \qquad (27)$$

Finally, we used the NORMRND facility in MATLAB5(1999) to estimate the white noise process e_t distributed with mean=0 and $\hat{\sigma}_e^2$ =1 11 as well as the process a_t distributed with mean=0 and $\hat{\sigma}_a^2$ =3.33. See Box and Jenkins (1976) for example

We then modeled the estimated error process b_t as

$\hat{b}_t = e_t + 0.84b_{t-1}$, t=1, 2 1400

The values of the process b_t is obtained using the recursion formula

$$\hat{b}_t = \sum_{i=0}^{t-1} \phi^i e_{t-i}, \quad t = 1,2,\ldots, \text{ as in (23)}$$

Ten values from t=201 to 210 are recorded in table1 as $\hat{b}_t$.

We also model the process of interest $\hat{\varepsilon}_t^2$ as

$\hat{\varepsilon}_t^2 = 0.12 + 0.75\hat{\varepsilon}_{t-1}^2 + a_t$, t=1,2....1400

The values of the process $\hat{\varepsilon}_t^2$ is obtained using the recursion formula

$$\varepsilon^2 = \sum_{i=0}^{t-1} \alpha^i (\alpha_0 + a_{t-i}), \quad t = 1,2,\ldots, \quad \text{as in (22)}$$

Ten values from t=201 to 210 are recorded in table1 as $\hat{\varepsilon}_t^2$.

Table 1: Simulated Processes ε_t^2 and g^2 and Estimated Processes $\hat{\varepsilon}_t^2$ and $\hat{b}_t$

SIMULATED PROCESSES		ESTIMATED PROCESSES	
ε_t^2	g_t^2	$\hat{\varepsilon}_t^2$	$\hat{b}_t$
2.237734	3.27867	2.4870	0.829662
3.011512	3.17412	2.9855	0.395716
0.993451	2.425333	1.1327	0.510802
2.732004	3.242643	2.7054	1.035373
2.799315	2.132874	2.7925	-0.34009
1.220803	1.05731	1.1563	-0.19107
4.928751	2.86533	5.8052	-2.3655
4.901011	2.43903	4.772	-2.52942
2.703241	1.78231	2.4086	-1.61641
3.365471	2.05712	3.1891	-1.71089

Examining Table1, we notice that the estimated process $\hat{\varepsilon}_t^2$ (estimated through g_t^2) is very close to the true (simulated) process ε_t^2.Also, the sum of the estimated process $\hat{\varepsilon}_t^2$ and the error process $\hat{b}_t$ is close to the observe process g_t^2 This shows that the process developed in this paper has performed well.

5.0 CONCLUSION

We developed a method which enables us to estimate both the ARCH (1) process and the AR(1)error process for ARCH (1) process corrupted with AR (1) errors. Simulation studies showed that the method performed very well

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