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AN ITERATIVE SCHEME FOR ESTIMATING THE PARAMETERS OF AN AUTOREGRESSIVE PROCESS

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ABSTRACT

In this paper, an iterative search method for estimating the parameters of an autoregressive process of order p, AR(p) is considered. The iterative search method was achieved by employing the Gauss-Jacobi iterative search method for solving linear system of equations. The estimation procedure includes finding the lag coefficients and the maximum lag order p of the process using the order selection method of the Akaike information criterion (AIC). The iterative search method is computationally efficient and converges to the same parameter estimate using the least square method. It has the advantage of handling large system of equations which is difficult to handle when using the Least squares method. Some simulated data and one real life data are used to demonstrate this approach.

KEY WORDS: Akaike information criterion; Autoregressive process of order p (AR(p)); Gauss-Jacobi.

1 INTRODUCTION

In order to estimate the parameters of an AR(p) model, it is appropriate to estimate the maximum order p of the process and the coefficients of the lag order of the process. If an order lower than the true order of the process is selected the estimate of the parameters will not be consistent, if higher order is selected the variance increases. The identification of the order of a stationary Box-Jenkins time series model has been crucial in the literature. Three main approaches have been proposed in the identification procedure of an AR(p) process. The first one is called the Box-Jenkins approach whose identification procedure is based on the study of the sample autocorrelation function (SACF) and the sample partial autocorrelation function (SPACF): Box and Jenkins (1976) and Bowerman and O'Connell (1993). The second approach is the information criterion procedure which involves the use of an order selection based on the minimisation of some given functions expressed in terms of the order p of the model. Example references are given in Shibata (1976), Hannan (1980) and Tsay (1984). The third is the procedure due to Pukkila et al. (1990), which is based on an autoregressive order determination criterion and on linear estimation methods. A purpose of this work therefore, is to present an alternative way of estimating the order and the lag coefficients of an AR(p) process using the iterative scheme of the Gauss-Jacobi search method for solving linear system of equations. This method has an advantage of choosing the order and estimating the parameters of the model simultaneously.

The rest of the paper is organized as follows: in section 2, the methodology is described; in section 3, some simulation results are given; in section 4, some computational experiments and results are presented while section 5 concludes the work

2 METHODOLOGY

Consider a stationary time series $\{y_t\}$ which satisfies the following linear equation for some integer p ≥ 1,

$$y_t = \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \cdots + \alpha_p y_{t-p} + \varepsilon_t \qquad (2.1)$$

where $\alpha_1, \alpha_2, \cdots, \alpha_p$ are real parameters such that the solution to the equation $y^p = \sum_{j=1}^{p} \alpha_j y^{p-j}$ are all less than 1 in absolute value and ε_t constitute a sequence of independent random variables with the same normal distribution $N(0, \sigma_\varepsilon^2)$ The model as presented by (2.1) is non-degenerate of order p if $\alpha_p \neq 0$ and of zero order if $y_t = \varepsilon_t$ i.e. the series is completely described by a white noise process: (Shibata) 1976. The presentation of (2.1) is based on the assumption that the mean level of the stationary process is $\mu = 0$, otherwise (2.1) can be re-written by subtracting μ from all the lag variables, $y_{t-j}, j = 1,2,\cdots, p$ From (2.1), if we assume that the true order of the process is known and is of order p, then the parameters can be estimated by considering the sum of squares error as:

$$\sum_{t=p+1}^{n} \varepsilon_t^2 = \sum_{t=p+1}^{n} \{y_t - \alpha_1 y_{t-1} - \alpha_2 y_{t-2} - \cdots - \alpha_p y_{t-p}\}^2 \qquad (2.2)$$

where ε_t is not defined for all t ≤ p Now Let us write the sum of squares error above as

$$\Delta = \sum_{t=p+1}^{n} \{y_t - \sum_{j=1}^{p} \alpha_j y_{t-j}\}^2 \qquad (2.3)$$

Then the least square estimate of α_j, that satisfy (2.3) is found by solving the set of equations

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$$\frac{\partial \Delta}{\partial \alpha_j} = 0, \qquad j = 1,2,\cdots,p \tag{2.4}$$

which lead us to considering a p system of equations in p unknowns of the form.

$$\begin{aligned}
\sum_{t=2}^{n} y_t y_{t-1} &= \alpha_1 \sum_{t=2}^{n} y_{t-1}^2 + \alpha_2 \sum_{t=3}^{n} y_{t-1} y_{t-2} + \cdots + \alpha_p \sum_{t=p+1}^{n} y_{t-1} y_{t-p} \\
\sum_{t=3}^{n} y_t y_{t-2} &= \alpha_1 \sum_{t=3}^{n} y_{t-1} y_{t-2} + \alpha_2 \sum_{t=3}^{n} y_{t-2}^2 + \cdots + \alpha_p \sum_{t=p+1}^{n} y_{t-2} y_{t-p} \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
\sum_{t=p+1}^{n} y_t y_{t-p} &= \alpha_1 \sum_{t=p+1}^{n} y_{t-1} y_{t-p} + \alpha_2 \sum_{t=p+1}^{n} y_{t-2} y_{t-p} + \cdots + \alpha_p \sum_{t=p+1}^{n} y_{t-p}^2
\end{aligned} \tag{2.5}$$

Now, let us denote the sum of cross product and sum of squares of the observations by the following notation.

$$g_{ij} = \sum_{t=p+1}^{n} y_{t-i} y_{t-j}, \qquad i = 0,1,\cdots,p \text{ and } j = 1,2,\cdots,p \tag{2.6}$$

Making use of the notation in (2.6), our system of equations presented in (2.5) can be written in matrix form as:

$$G\alpha = g_0 \tag{2.7}$$

where $$G = \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1p} \\ g_{21} & g_{22} & \cdots & g_{2p} \\ \vdots & \vdots & & \vdots \\ g_{p1} & g_{p2} & \cdots & g_{pp} \end{pmatrix} \tag{2.8}$$

$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_p)'$, $g_0 = (g_{01}, g_{02}, \cdots, g_{0p})'$. G is symmetric, since $g_{ij} = g_{ji} \ \forall \ i \neq j$. G is diagonally dominant, it includes $g_{ii} > g_{kk} \ \forall \ i < k$ Since G is non-singular, the solution to equation (2.7) is given by $\hat{\alpha} = G^{-1} g_0$.

Now, for large P system of equations, computation of G^{-1} becomes cumbersome. In fact, the entries $g_{ij} \forall i, j$ involves sums of squares and sums of cross product of the lag variables, which requires heavy computations. In other to save space, we require that the system of equations presented in (2.5) can be re-written compactly as:

$$\sum_{t=j+1}^{n} y_t y_{t-j} = \alpha_j \sum_{t=j+1}^{n} y_{t-j}^2 + \sum_{t=m+1}^{n} \sum_{\substack{i=1 \\ i \neq j}}^{p} \alpha_i y_{t-i} y_{t-j}, \ (i,j) = 1,2,\ldots,p; m = \max(i,j) \tag{2.9}$$

hence we could achieve $\alpha_{j's}$ by developing an iterative method from (2.9) in the following manner below:

$$\alpha_j^{(s+1)} = \frac{\left(\sum_{t=j+1}^{n} y_t y_{t-j} - \sum_{t=m+1}^{n} \sum_{\substack{i=1 \\ i \neq j}}^{p} \alpha_i^{(s)} y_{t-i} y_{t-j} \right)}{\sum_{t=j+1}^{n} y_{t-j}^2}, (i,j) = 1,2,\ldots,p; m = \max(i,j) \tag{2.10}$$

Equation (2.10) is actually the Gauss-Jacobi iterative scheme for solving linear system of equations, as in Burden and Faires (1993, pp. 406-408). This scheme converges if the system of equations is of full rank and if it is non-singular. The stopping criteria requires $\left| \alpha_j^{(S+1)} - \alpha_j^{(S)} \right| < Tol$, $j = 1,2,\cdots,p$ and $s = 0,1,\cdots,L$, where Tol is called the error tolerance allowed for the iterative scheme to stop and it is usually chosen by the user The initial guess $\alpha_j^{(0)}, j = 1,2,\cdots,p$ must be chosen to ensure fast convergence. In this work, the initial guess chosen was $\alpha_j^{(0)} = 0, j = 1,2,\ldots,p$ and the error tolerance, $Tol = 10^{-6}$. However, fast convergence is guaranteed if $\alpha_j^{(0)}, j = 1,2,\cdots,p$ are chosen close to the true parameters $\hat{\alpha}_{js}$ provided they satisfies stationarity condition of an AR(p) model.

Moreover, the AIC requires the maximum order p for which the function $AIC(\alpha) = n\log\sigma_\varepsilon^2(p) + 2p$ attains its minimum, where $\sigma_\varepsilon^2(p)$ is the mean sum of squares of the residuals computed from (2.2) after estimating the parameters of the model for any given order p>0 (an integer): Shibata (1976). Alternatively, Sen Liew and Chong (2005), also defined the AIC as $AIC(\alpha) = -n\log\sigma_\varepsilon^2(p) + 2p$ whenever $\log\sigma_\varepsilon^2(p) < 0$

3 Simulation results

In order to demonstrate the procedure described in section 2, the following data generating process (DGP) are considered.

Model (a) $y_t = 0.5y_{t-1} + \varepsilon_t, y_0 = -0.5$

Model (b) $y_t = 0.2602y_{t-1} - 0.2578y_{t-2} + \varepsilon_t, y_0 = -0.5$

Model (c) $y_t = 0.35y_{t-1} - 0.125y_{t-2} + 0.25y_{t-3} + \varepsilon_t, y_0 = 0.5$

The error, ε_t , which is normally distributed with mean 0 and variance 1 is generated using the Microsoft Excel data analysis library, for 500 observations, respectively. The first 300 simulated observations are discarded to remove initialization effect from the data and therefore the remaining 200 simulated data is used for our experiment.

4 Computational Experiment and Results

As an illustration, each of the simulated time series data is subjected to an AR(p) models of order 1 up to order 3 to see if the true order p of the process can be captured by the minimum AIC using the iterative scheme. Similarly, the parameters of each of the model are obtained using the matrix in (2.7) that was derived from the least square method. The results obtained using matrix (2.7) will be called the direct method and the results from the iterative scheme given in (2.10) will be called iterative method.

4.1 Direct method

When p = 1, the estimate of the parameter, α_1, using matrix (2.7) when the order is 1 gives the following:

$$\hat{\alpha}_1 = \frac{g_{01}}{g_{11}} \tag{4.1.1}$$

When p = 2,

$$\begin{pmatrix} g_{01} \\ g_{02} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \text{ with } g_{12} = g_{21} \tag{4.1.2}$$

The estimate of the parameters α_1, α_2 using matrix (4.1.2) when the order is 2 gives the following:

$$\hat{\alpha}_1 = \frac{g_{01}g_{22} - g_{02}g_{12}}{g_{11}g_{22} - g_{12}^2},\ \hat{\alpha}_2 = \frac{g_{02}g_{11} - g_{01}g_{12}}{g_{11}g_{22} - g_{12}^2}$$

When p = 3

$$\begin{pmatrix} g_{01} \\ g_{02} \\ g_{03} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \text{ with } g_{12} = g_{21},\ g_{13} = g_{31} \text{ and } g_{23} = g_{32} \tag{4.1.3}$$

The estimate of the parameters $\alpha_1, \alpha_2, \alpha_3$, using matrix (4.1.3) when the order is 3 gives the following:

$$\hat{\alpha}_1 = \frac{g_{01}(g_{22}g_{33} - g_{23}^2) + g_{02}(g_{13}g_{23} - g_{12}g_{33}) + g_{03}(g_{12}g_{23} - g_{13}g_{22})}{g_{11}(g_{23}g_{33} - g_{23}^2) + 2g_{12}g_{13}g_{23} - g_{12}^2g_{33} - g_{13}^2g_{22}}$$

$$\hat{\alpha}_2 = \frac{g_{01}(g_{13}g_{23} - g_{12}g_{33}) + g_{02}(g_{11}g_{33} - g_{13}^2) + g_{03}(g_{12}g_{13} - g_{11}g_{23})}{g_{11}(g_{23}g_{33} - g_{23}^2) + 2g_{12}g_{13}g_{23} - g_{12}^2g_{33} - g_{13}^2g_{22}}$$

$$\hat{\alpha}_3 = \frac{g_{01}(g_{12}g_{23} - g_{13}g_{22}) + g_{02}(g_{12}g_{13} - g_{11}g_{23}) + g_{03}(g_{11}g_{12} - g_{12}^2)}{g_{11}(g_{23}g_{33} - g_{23}^2) + 2g_{12}g_{13}g_{23} - g_{12}^2g_{33} - g_{13}^2g_{22}}$$

The results of the parameter estimates of the above procedure are given in Table1 below:

Table1: The summary statistics of the estimated parameters of our simulated model in section 3 using the direct method

Models	Order(p)	Parameter(s)	Variance(p)	AIC
Model (a)	1	$\alpha_1 = 0.4944$	0.9730	7.4806
Model (b)	2	$\alpha_1 = 0.2331, \alpha_2 = -0.2720$	0.9732	9.4250
Model (c)	3	$\alpha_1 = 0.3558, \alpha_2 = -0.1614, \alpha_3 = 0.3145$	0.9739	11.2852

4.2 Iterative method

The implementation of the iterative scheme presented in (2.10), was achieved using a program written in Microsoft QuickBasic. The algorithm of the program is given as follows:

Step 1: Generate the error sequence, ε_t.

Step 2: Generate artificial time series observations for each of the model in section 3 using the error sequence, ε_t, in step 1.

Step 3: Estimate the parameters of the model using the iterative scheme of (2.10) by setting p =1 to 3. At each point of convergence for the specific parameter(s), estimate the AIC.

Step 4: The order p and the given parameters at which the process attains its minimum value gives the model.
The estimates of the parameters obtained from the iterative scheme are displayed in Tables 2, 3 and 4, respectively.

Table 2: The summary statistics of the parameter estimate of the three models with their corresponding error variance and the AIC

Order(p)	Parameter (s)	Variance (p)	AIC	Iterations
1*	$\alpha_1 = 0.4944$	0.9730	7.4806	1
2	$\alpha_1 = 0.4923, \alpha_2 = 0.0043$	0.9740	9.2676	15
3	$\alpha_1 = 0.4918, \alpha_2 = -0.0295, \alpha_3 = 0.0691$	0.9741	11.2671	58

For true order P = 1, n =200, the true model is given by: $y_t = 0.5y_{t-1} + \varepsilon_t$

Table 3: The summary statistics of the parameter estimate of the three models with their corresponding error variance and the AIC

Order (p)	Parameter (s)	Variance (p)	AIC	Iterations
1	$\alpha_1 = 0.1827$	1.0515	12.0451	1
2*	$\alpha_1 = 0.2331, \alpha_2 = -0.2720$	0.9732	9.4250	6
3	$\alpha_1 = 0.2521, \alpha_2 = -0.2884, \alpha_3 = 0.0705$	0.9727	11.5372	11

For true order P = 2, n =200, the true model is given by: $y_t = 0.2602y_{t-1} - 0.2578y_{t-2} + \varepsilon_t$

Table 4: The summary statistics of the parameter estimate of the three models with their corresponding error variance and the AIC

Order (p)	Parameter (s)	Variance (p)	AIC	Iterations
1	$\alpha_1 = 0.3219$	1.0768	16.7955	1
2	$\alpha_1 = 0.3399, \alpha_2 = -0.0557$	1.0767	18.5649	8
3*	$\alpha_1 = 0.3558, \alpha_2 = -0.1614, \alpha_3 = 0.3145$	0.9739	11.2852	14

For true order P = 3, n =200, the true model is given by: $y_t = 0.35y_{t-1} - 0.125y_{t-2} + 0.25y_{t-3} + \varepsilon_t$

4.3 RESULTS

Table 1 represents the estimate of the parameter(s) of each of the simulated models where the true order is assumed known and fixed. Then, Tables 2, 3 and 4 represent the results obtained from the iterative scheme of (2.10). The asterisks on some values of the order (p) indicate the point for which the AIC attains its minimum value. From Tables 2, 3 and 4, it is clearly evident that the true parameters of the model using the iterative search method converges to the same parameter(s) values obtained using the least squares method.

4.4 DATA

The data used here is taken from Bowerman and O'Connell (1993, pp. 471). The data comprise the daily readings of the viscosity of a chemical product, XB-77-5, for 95 days. The viscosity data as analysed by these authors, was found to be stationary since the SACF dies down fairly quickly in a damped sine-wave fashion and the SPAC has a spike at lags 1 and 2 and cuts off after lag 2. As a result of this, they came up with an AR(2) model for the viscosity data. Representing their model as

$y_t = \delta + \alpha y_{t-1} + \alpha_2 y_{t-2} + \varepsilon_t$ where $\delta = \mu(1 - \alpha_1 - \alpha_2)$, the least squares point estimates of the parameters yielded $\hat{\delta} = 26.8577, \alpha_1 = 0.61356, \alpha_2 = -0.38304$ Writing this model as $(y_t - \mu) = \alpha_1(y_{t-1} - \mu) + \alpha_2(y_{t-2} - \mu) + \cdots + \alpha_p(y_{t-p} - \mu) + \varepsilon_t$ where $\mu = n^{-1}\sum_{t=1}^{n} y_t$ is the sample mean, p is unknown and as usual $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$ The result of the implementation of the iterative scheme presented in (2.10) to the above situation, is given in Table 5 below

Table 5: The summary statistics of the estimated parameters of the viscosity data with their corresponding error variance and the AIC

Order (p)	Parameter (s)	Variance (p)	AIC	Iterations
1	$\alpha_1 = 0.4470$	4.6992	149.0020	1
2*	$\alpha_1 = 0.6130$ $\alpha_2 = -0.3836$	3.8754	131.5490	12
3	$\alpha_1 = 0.5507$ $\alpha_2 = -0.2822$ $\alpha_3 = -0.1682$	3.7493	132.6907	23
4	$\alpha_1 = 0.5459$ $\alpha_2 = -0.2896$ $\alpha_3 = -0.1526$ $\alpha_4 = -0.0297$	3.7162	132.7078	17

From Table 5, it is clear that the iterative scheme was able to capture the true order of the process and the corresponding parameters with the slight difference resulting from the mode of presenting the model.

5 CONCLUSION

An iterative method for estimating the parameters of an AR(p) process has been considered. The procedure employs the Gauss-Jacobi search method for solving linear systems of equations. This procedure is found to be as efficient as the Least squares method in estimating the parameters of an AR(p) process when the order p is already known. The implementation requires a simultaneous estimate of the order p and the parameters $\alpha_j, j = 1,2,\cdots, p$ It has the advantage of handling large system of equations which is difficult to handle when using the Least squares method

REFERENCES

Bowerman, B. L. and O'Connell, R. T., 1993. Forecasting and Time Series. An applied approach. 3rd Edition. Duxbury Press, California.

Box, G. E. P and Jenkins, G M., 1976 Time series analysis, forecasting and control. Holden-Day, San Francisco.

Burden, R. L. and Faires, J D . 1993 Numerical Analysis. Fifth Edition, PWS publishing company, Boston.

Hannan, E. J., 1980 The estimation of the order of an ARMA process, The Annals of Statistics, Vol. 8, No. 5, 1071-1081

Kendall, M. and Ord, J. K., 1990. Time Series, Third Edition, Edward Arnold, London.

Sen Liew, V K. and Chong, T T L., 2005. Autoregressive lag length selection criteria in the presence of ARCH Errors, Economics bulletin, vol.3, No. 19, 1-5.

Pukkila, T., Koreisha, S. and Kallinen, A., 1990. The identification of ARMA models, Biometrika, 77, 3, 537-48.

Shibata, R., 1976. Selection of the order of an autoregressive model by Akaikes information criterion, Biometrika 63, 1,117-26.

Tsay, R. S., 1984. Order selection in nonstationary autoregressive models. Vol 12, No. 4, 1425-1433.

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ERROR AND PROCESS ESTIMATION OF ARCH (1) MODEL CORRUPTED BY AR(1) PROCESS

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ABSTRACT

We showed how autocovariance functions can be used to estimate the ARCH(1) process corrupted by AR(I) errors. we performed simulation studies to demonstrate our findings. The studies showed that our model was able to very closely estimate the required ARCH process in the presence of AR(1) errors.

KEY WORDS: AR, ARMA, ARCH, Error and Process Estimation

1.0 ARCH FRAME WORK.

Let $\{y_t\}$ denote a stochastic process with mean μ_t, then the error term is defined as (see Bollerslev, Engle and Nelson (1994))

$\varepsilon_t = y_t - u_t$

Under the assumption of constant variance, and correct model specification,
ε_t will be distributed as Z_t where Z_t is any symmetric distribution. However, under a time- varying variance condition, ε_t will be expressed as a product, ie

$\varepsilon_t = Z_t\ h_t^{1/2}$

where h_t is the conditional variance at time t and Z_t is any symmetric distribution. Bollerslev, Engle and Nelson (1994) defines the ε_t process to follow an Autoregressive Conditional Heteroscedascity (ARCH) model ARCH process if

$E_{t-1}(\varepsilon_t)=0 \quad t=1,2,$

In addition, the conditional variance is

$$h_t = \mathrm{var}_{t-1}\{\varepsilon_t\} = \mathrm{E}_{t-1}\{\varepsilon_t^2\},$$

where $\mathrm{E}_{t-1}(.)$ denotes the conditional expectation when the conditioning set is compose of information up to time t-1

Engle's (1982) ARCH(q) model is presented as ARCH model as a linear function of the past squared disturbances. That is

$$\varepsilon_t^2 = z_t^2 h_t,$$

and

$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2,$$

2.0 PROBLEM FORMULATION

Consider the ARCH (1,1) model equation

$$h_t = \alpha_0 + \alpha\varepsilon_{t-1}^2, \qquad (1)$$

and

$$\varepsilon_t^2 = Z_t^2 h_t,$$

with parameter constraints

$$\alpha_0 > 0, \alpha \geq 0$$

These constraints are meant to ensure that the variance is positive.
Equation (1) admits transformation to AR (1) model through the substitution

$$a_t = \varepsilon_t^2 - h_t,$$

to get

$$\varepsilon_t^2 = \alpha_0 + \alpha\varepsilon_{t-1}^2 + a_t, \qquad (2)$$

or

$$(1 - \alpha L)\varepsilon_t^2 = \alpha_0 + a_t \qquad (3)$$

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