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ON THE STABILITY OF EVOLUTION EQUATIONS

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ABSTRACT

A Quasi-Linear Hyperbolic Evolution problem in a Banach space with m-accretive operator is considered and conditions which guarantee asymptotic stability of its solution in a dense subset of the space are given.

KEY WORDS: Quasi-Linear, Hyperbolic, Evolution Problem

1.0 INTRODUCTION

An evolution equation as given below will be considered

$$\frac{du}{dt} = Au(t) + f(u,t) \tag{1.1}$$

where $f(u,t): X \times R_+ \to X$, X a Banach Space. Merenkov (1993), investigated the stability for this evolution equation using Lyapunov's functionals where $f(u,t): X \times R_+ \to X$ is continuous with respect to u for almost all $t \in R_+$ and strongly measurable with respect to t for all $u \in X$, X a Banach space with dual space X^*, inner product $(.,.)$ and norm $\|.\|_X$, with the operator A the generator of a strongly continuous semigroup T( ) of bounded linear operators on X. Let $u_t(s) = u(t+s), s \in [0,\gamma], \lambda > 0$ be the section t of the function u.

Definition 1 (Merenkov 1993; Kartsators and Parrott 1982): The zero solution of equation (1.1) is said to be Lyapunov stable if, for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $(u,t) \in X \times R_+, \|u(0)\|_X < \delta \Rightarrow \|u(t)\|_X < \varepsilon$.

Definition 2 (Merenkov 1993; Kartsators and Parrott 1982): The zero solution of equation (1 1) is said to be asymptotically stable if it is Lyapunov stable and in addition, there exists $h > 0$ such that for all $u \in X, \|u(0)\|_X < h \Rightarrow \lim_{t \to r} \|u(t)\|_X = 0$

Merenkov (1993) established that if this evolution equation referred to above is constructed from a parabolic or a hyperbolic partial differential equation, then the semigroup S(.) usually satisfies the condition; for all $u \in X$ there exists $M, w \in R_+$ such that $\|S(t)u\|_X \le Me^{-wt}\|u\|_X$ Also if w > 0 and $f(u,t)$ is a bounded function, then it is possible to establish asymptotic stability of the zero solution using

$$u(t) = S(t-t_0)u(t_0) + \int_{t_0}^{t} S(t-t_0)f(u(s),s)ds \tag{1.2}$$

Egwurube and Garba (2003), considered a quasi-linear hyperbolic differential equation which was transformed into the evolution equation;

$$\frac{du}{dt} + Au(t) = 0, u(0) = u_0 \tag{1.3}$$

defined on a Banach space $L^1[0,1]$ with D(A) = C[0,1] and proved that the operator A is m-accretive and that it does admit a solution This paper considers this quasi-linear hyperbolic evolution problem and gives conditions which guarantee asymptotic stability of its solutions in C[0,1]; a dense subset of the Banach space.

2.0 MAIN RESULTS

Let u^* be the steady state solution of (1 3) above and assume $x = u - u^*$ so that

$$\frac{dx}{dt} + A\, x(t) = h(x), x(0) = x_0 \tag{2.2}$$

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where A is a bounded linear operator and $h(x)$ represents the non-linear term. By Ladeira and Tanaka (1997) and the method of variation of parameter (Boyce and DiPrima, 1977), the solution of (2.2) is

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}h(x(\tau))d\tau \tag{2.3}$$

Comparing (2.3) with (1.2) and observing that exp(*A t*) is a strongly continuous semigroup, we proceed to establish asymptotic stability of the zero solution of (1.3).

Lemma 1 (Liadi, 2003): Let A be a continuous linear operator on a Banach space E then there exists a bounded operator $(\xi I - A)^{-1}, I$ being the identity operator if $|\xi| > \| A \|$ and $(\xi I - A)^{-1} = \sum_{n=1}^{\infty} \frac{1}{\xi^{n+1}} A^n$ with a unique resolvent $H^* = \left(\frac{1}{\xi} + o\left(\frac{1}{\xi^2}\right)\right)\delta \ as \ |\xi| \to \infty$.

Applying Laplace transform to (2.3) with respect to *t* gives

$$x(\xi) = (\xi I - A)^{-1}\{x_0 + h(x(\xi))\}.$$

where $x(\xi)$ is the Laplace transform of *x(t)* and $h(x(\xi))$ is the Laplace transform of *h(x(t))*.

Theorem 2: Suppose (i) $|h(x(\xi))| \le \delta|x(\xi)|, \delta > 0$

(ii) that the unique resolvent of $(\xi I - A)^{-1}\delta$ exists when Re $\xi < 0$

(iii) that there exists $\delta > 0$ such that $\|x(0)\| < \delta$

then the solution of the evolution equation (1.3) is asymptotically stable on C[0,1].

Proof: $|x(\xi)| \le |(\xi I - A)^{-1}|\{|x_0| + |h(x(\xi))|\}$

Suppose $|h(x(\xi))| \le \delta|x(\xi)|, \delta > 0$ then $|x(\xi)| \le |(\xi I - A)^{-1}|\{|x_0| + \delta|x(\xi)|\}$.

Hence $|(I - (\xi I - A)^{-1}\delta||x(\xi)| \le |(\xi I - A)^{-1}||x_0|$. Suppose for Re $\xi > \alpha, \alpha < 0; (\xi I - A)^{-1}$ exists and suppose further that the unique resolvent of $(\xi I - A)^{-1}\delta$ exists when Re $\xi < 0$ and can be represented by H^*.

Thus $|x(\xi)| \le |(\xi I - A)^{-1}||x_0| + H^*(|\xi I - A)^{-1}||x_0|)$.

Taking the inverse Laplace transform of the above and denoting the inverse Laplace transform of H^* by $\widetilde{H}^*$, where

$$\widetilde{H}^* = X(t+\gamma) \ where \ X(t+\gamma) = \begin{cases} 0, t < \gamma \\ 1, t \ge \gamma \end{cases}$$

Suppose also that $T(\xi) = (\xi I - A)^{-1}$ then $T(\xi) = \xi^{-1}I + o(\xi^{-2}) \ as \ |\xi| \to \infty, \alpha \le \text{Re}\,\xi < 0$.

Therefore $T(t) = O(e^{\alpha t}) \ as \ t \to \infty$ and

$$|x(t)| \le e^{\alpha t}|x_0|\left(1 + \int_{-\infty}^{t} e^{-\alpha\tau}X(\tau+\gamma)d\tau\right)$$

$$= e^{\alpha t}|x_0|\left(1 + \int_0^{\gamma} e^{-\alpha\tau}d\tau\right) = e^{\alpha t}|x_0|\left(1 + \frac{1 - e^{-\alpha\gamma}}{\alpha}\right), t \ge 0$$

so that in C[0,1] $\lim_{t\to\infty}\|x(t)\| = 0$. Hence the solution is asymptotically stable in C [0, 1].

3.0 CONCLUSION

Thus the solutions of the quasi-linear evolution problem in a dense subset of $L^1[0,1]$, are bounded in the neighbourhood of the origin and converge to the origin as *t* approaches infinity.

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