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AN APPLICATION OF DARBO'S FIXED POINT THEOREM IN THE RELATIVE CONTROLLABILITY OF NON-LINEAR SYSTEMS

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ABSTRACT

Sufficient conditions for the relative controllability of a class of nonlinear systems with distributed delays in the control are established. Our results are based on the measure of non-compactness of a set and the Darbo's fixed point theorem.

KEYWORD: Controllability, relative controllability, nonlinear systems, implicit derivative

1. INTRODUCTION

Quite unlike linear systems where standard techniques of investigating their controllability have been sufficiently developed, computational methods of investigating the controllability of non-linear systems have remained elusive. However, the linearization methods have been in use and the fixed point approach has also proved useful, see Klamka 1980. Various fixed point theorem have been applied in the controllability of nonlinear analysis, some of the fixed point theorems commonly applied includes, the Schauder's fixed point theorem, see Balachandran and Dauer 1996, Klamka 1980; the Nussbaum's fixed point theorem, see Balachandran and Park 2003, Balachandran and Anandhi 2004, Kwun et al 1991; the Schaefer fixed point theorem see Atmania and Mazouzi 2005, Balachandran and Sakthivel 2000; the Sadoski's fixed point theorem, see Anichini et al 1986, Fu 2003 etc. From these studies one readily sees the difficulty of converting a given control system to an operator equation where the operator is well defined in its domain; and endowed with the smoothness properties that will make it satisfy the set of conditions for the application of a fixed point theorem. For systems with implicit derivative the problem is made more complex with the difficulty of making the right choice of an appropriate state space. However Balachandran 1987, Davies 2005 and Decca 1981 have illustrated the use of Darbo's fixed point theorem in the investigation of the controllability of discrete systems with implicit derivative based on the measure of non-compactness of a set, but not with distributed delays in the control. Our interest in this research therefore is to find the relative controllability of a class of nonlinear systems with distributed delays in the control and having implicit derivative of the form

$$\dot{x}(t) = A(t, x(.), \dot{x}(.))x(.) + \int_{-h}^{0} d_s H(t, s, x(.))u(t+s) + f(t, x(.), \dot{x}(.), u(.))$$

by using the Darbo's fixed point theorem .Our result shall incorporate and extend that of Decca 1981, Balachandran 1987 and that in Klamka 1980.

2. BASIC NOTATIONS AND PRELIMINARIES

Let $E = (-\infty, \infty)$ and E^n be the n-dimensional Euclidean space with norm $|\cdot|$. Let $C = C([-h, h], E^n)$ denote the space of continuous functions mapping the interval $[-h, 0] \in E^n$ into E^n, $h > 0$ with the suprenum norm $\|\cdot\|$ defined by

$\|\phi\| = \sup|\phi(\theta)|;\ \phi \in C,\ -h \le \theta \le 0$. While $C' = C'([-h, 0], E^n)$ denotes the space of differentiable functions mapping the internal $[-h, 0]$ into E^n.

Let $(X, \|\cdot\|)$ be a Banach space and Q a bounded subset of X, the measure of non-compactness of Q is given by $\mu(Q) = \inf\ \{r \ge 0$: Q can be covered by a finite balls of radii less than r}. For the space of continuous functions $C([t_0, t_1], E^n)$, the measure of non-compactness of the set Q is given by

$$\mu(Q) = \frac{1}{2} W_0(Q) = \frac{1}{2} \lim_{h \to 0} W(Q, h)$$

where $W(Q, h)$ is the common modulus of continuity of the functions which belong to the set Q, that is;

$$W(Q, h) = \sup_{x \in Q} \{\sup |x(t) - x(s)| : |t - s|\} \le h$$

For the space of differentiable functions $C'([-h, 0], E^n)$ we have

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$\mu(Q) = \frac{1}{2}W_0(DQ)$ where $DQ = \{\dot{x} : x \in Q\}$

If $t \in [t_0, t_1]$ we let $x(t) \in C'$ be defined for $s \in [-h, 0]$. Also, for functions $u : [t_0 - h, t_1] \to E''$, $h > 0$ and $t \in [t_0, t_1]$ then u_t denotes the functions on $[-h, 0]$ defined by $u_t(s) = u(t+s)$ for $s \in [-h, 0]$. The integral is in the Lebesgue-Steiltjes sense.

Throughout the sequel, the controls of interest are

$B = L_\infty([t_0, t_1], E'')$ and $U \subseteq L_\infty([t_0, t_1], E'')$ a closed and bounded subset of B with zero in its interior relative to B

We consider the nonlinear dynamical system with distributed delays in control with implicit derivative in the state and perturbation given by

$$\dot{x}(t) = A(t, x(t), \dot{x}(t))x(t) + \int_{-h}^{0} d_s H(t, s, x(t))u(t+s) + f(t, x(t), \dot{x}(t), u(t)) \tag{2.1}$$

where the $n \times m$ matrix valued function $H(t, s, x(t))$ is continuous in (t, x) for each fixed s and is of bounded variation in s on $[-h, 0]$ for each $(t, x) \in [t_0, t_1] \times C'$ and $A(t, x(t), \dot{x}(t))$ is an $n \times n$ matrix function measurable in $(t, x) \in E \times E$ normalized so that

$A(t, x(t), \dot{x}(t)) = 0;\ x \geq 0$

$A(t, x(t), \dot{x}(t)) = A(t-h)\ \ for\ \ s \leq -h$

There is an integrable function m such that $|A(t, x(t), \dot{x}(t))x(t)| \leq m(t)\|x(t)\|$ for $t \in (-\infty, \infty)$, $x(t) \in C''$.

The function f is continuous and satisfies the Lipschitz condition in all its arguments. Replacing the argument $x(t)$ by $z \in C'[t_0, t_1]$ in the matrices A *and* H, the system (2.1) is linear and we can deduce the variation of parameter by direct integration, and is of the form

$$x(t_1) = X(t, t_0, z, \dot{z})x(t_0) + \int_{t_0}^{t} X(t, t_0, z, \dot{z})\left(\int_{-h}^{0} d_s H(l-s; z)u(l+s)dl\right)$$

$$+ \int_{t_0}^{t} X(t_1, l; z, \dot{z}) f(l, x(l), \dot{x}(l), u(l))dl \tag{2.2}$$

where $X(t, l; z, \dot{z})$ is the transition matrix of the linear system. $\dot{x}(t) = A(t, z(t), \dot{z}(t))x(t)$ satisfying $X(t, l; z, \dot{z}) = I_n$, where I_n is the identity $n \times n$ matrix. We consider the solution $x(t)$ of the system (2.1) for $t = t_1$, with the following assumption

$$H_t(l, s; z) = \begin{cases} H(l, s, z), & for\ \ l \leq t \\ 0, & for\ \ l > t \end{cases} \tag{2.3}$$

and is of the form

$$x(t_1) = X(t_1, t_0, z, \dot{z})x(t_0) + \int_{-h}^{0} dH_s\left(\int_{t_0+s}^{t_0} X(t_1, l-s, z, \dot{z})H(l-s, s; z)u_{t_0} dl\right)$$

$$+ \int_{t_0}^{t_1}\left(\int_{-h}^{0} X(t_1, l-s; z, \dot{z})dH_{t_1}(l-s, s, z)u(l)dl\right) + \int_{t_0}^{t_1} X(t_1, l; z, \dot{z}) f(l, x(l), \dot{x}(l), u(l))dl \tag{2.4}$$

where the symbol dH_s denotes that the integration is in the Lebesque-Stieltjes sense with respect to the variable s.

We now introduce the following notations for brevity

$$g(t_1) = g(y(t_0), x(t_1), z, \dot{z}) = x_1 - X(t_1, t_0, z, \dot{z})x(t_0)$$

$$+ \int_{-h}^{0} dH_S\left(\int_{t_0+s}^{t_0} X(t_1, l-s, z, \dot{z})H(l-s, s; z)u_{t_0} dl\right) - \int_{t_0}^{t_1} X(t_1, l; z, \dot{z}) f(l, x(l), \dot{x}(l), u(l))dl \tag{2.5}$$

we let

$$Z(t_0, l, s, z, \dot{z}) = \int_{-h}^{0} X(t_1, l-s; z, \dot{z})dH_{t_1}(l-s, s, z), u(l))dl \tag{2.6}$$

We define the $n \times n$ matrix for the system (2.1) at $t = t_1$ and is of the form

$$W(t_0,t_1,z,\dot{z}) = \int_{t_0}^{t_1} Z(t_0,l,s,z,\dot{z})Z^T(t_0,l,s,z,\dot{z})\,dl \tag{2.7}$$

where the symbol T denotes the matrix transpose.

Definition 2.1

The set $y(t) = \{x(t), u_t\}$ is said to be the complete state of the systems (2.1) in time t see Davies 2005.

Definition 2.2

Systems (2.1) is said to be relatively controllable on $[t_0,t_1]$, if for every initial complete state $y(t_0)$ and every $x_1 \in E^n$, there exists a control $u(t)$ defined on $[t_0,t_1]$ such that the corresponding trajectory of system (2.1) satisfies $x(t_1) = x_1$.

Definition 2.3 (DARBO'S FIXED POINT THEOREM)

If S is a non-empty, bounded, closed, convex subset of X and $P: S \to S$ is a continuous mapping such that for any $Q \subset S$, we have $\mu(PQ) \le k\mu(Q)$
where k is a constant $0 < k \le 1$, then P is a fixed point see Decca 1981.

3. RELATIVE CONTROLLABILITY RESULTS

Here we state and prove the theorem, which summarizes our result on the relative controllability of system (2.1).

CONDITIONS FOR RELATIVE CONTROLLABILITY

The following sufficient condition for relative controllability can be formulated, given system (2.1) with conditions as spelt out that, A, H and f are continuous functions in all their variables; and that

$$| A(t,z(t),\dot{z}(t))x(t) | \le m(t)\| x(t) \| \tag{3.1}$$

where $m(t)$ is an integrable function, $H(t,s;z(t))$ is of bounded variation in s on $[-h,0]$. The function f satisfies the Lipschitze condition with respect to the state variable, and therefore uniquely determined by any control. Furthermore,

$\| A(t,z(t),\dot{z}(t)) \| \le M$ for each $s \in [-h,0]$

$\| H(t,s;z(t)) \| \le N$ for each $s \in [-h,0]$

$\| f(t,x(t),\dot{x}(t),u(t)) \| \le k$ for each $t \in [t_0,t_1]$ (3.2)

$z \in C'$, $u \in C([t_0,t_1])$, where M, N and k are some positive constants and also, for every $x \in C'$, $u \in C$ and $t \in [t_0,t_1]$.

(d) $\| f(t,x(t),\dot{x}(t),u(t)) - f(t,x(l),\dot{x}(l),u(l)) \| \le k| x(t) - x(l)|$

where k is a positive constant such that $0 < k \le 1$

THEOREM 3.1

Assume that

$$\inf \det W(t_0,t_1,z,\dot{z}) > 0 \tag{3.3}$$

with $z \in C'$ then system (2.1) is relatively controllable on $[t_0,t_1]$.

PROOF

We define the control $u(t)$ for $t \in [t_0,t_1]$ as follows;

$$u(t) = Z^T(t_0,l,s,z,\dot{z})W(t_0,t_1,z,\dot{z})g(y(t_0,x(t_1),z,\dot{z})) \tag{3.4}$$

where $y(t_0)$ and $x(t_1) = x_1 \in E^n$ are chosen arbitrarily. The inverse of $W(t_0,t_1,z,\dot{z})$ is possible by condition (3.2). Substituting (3.4) into (2.4) to replace $u(t)$ and using (2.5) and (2.7). It is clear that the control $u(t)$ defined by (3.4) steers the initial complete state $y(t)$ to the final state $x(t_1) = x_1 \in E^n$. The actual substitution of (3.4) into (2.4) yields

$$x(t_1) = X(t_1, l, s, z, \dot{z})x(t_0) + \int_{-h}^{0} dH_s\left(\int_{t_0+s}^{t_1} X(t_1, l-s, z, \dot{z})H(l, s, s; z)\right)u_{t_0} dl$$
$$+ \int_{t_0}^{t_1}\left(\int_{-h}^{0} X(t_1, l-s, z, \dot{z})dH_t(l-s, s; z)\right)Z^T(t_0, l, s, z, \dot{z})W^{-1}(t_0, t_1, z, \dot{z})g(t_1)\,dl \tag{3.5}$$
$$+ \int_{t_0}^{t_1} X(t_1, l, z, \dot{z})f(l, x(l), \dot{x}(l), u(l))\,dl$$

We consider the right hand side of (3.5) as non-linear operator which maps the Banach space $C'([-h,0], E^n)$ into itself. Hence we can write (3.5) as $\dot{x}(t) = T(x)(t)$. This operator is continuous since all the functions involved in the operator are continuous. Define the closed, convex subset G by;

$$G = \{x : x \in C'([-h,0], E^n)\}, \; \|x\| \le N_1, \; \|Dx\| \le N_2 \tag{3.6}$$

where the positive real constants N_1 and N_2 are given by

$$N_1 = |x(t_0)|\exp M(t_1 - t_0) + a + (t_1 - t_0)b^2ck_1 + k(t_1 - t_0)\exp M(t_1 - t_0)$$
$$N_2 = MN_1 + bcNk_1k_2 + k$$
$$k_1 = |x_1| + |x(t_0)|\exp M(t_1 - t_0) + a + k(t_1 - t_0)\exp 2m(t_1 - t_0)$$
$$k_2 = Max\,Variation\; H(t, s, z(t)), \; t, s \in [-h, 0],$$
$$a = \sup renum\left\| \int_{-h}^{0} dH_s\left(\int_{t0+s}^{t_1} X(t_1, l-s, z, \dot{z})H(l-s, s; z)\right)u_{t0}dl \right\|, \; t \in [t_0, t_1]$$
$$b = Supremum. \| Z(t_0, l, s, z, \dot{z}) \|, \; z \in C'$$

$$c = Supremum \| W^{-1}(t_0, t_1, z, \dot{z}) \|$$

the constraints a, b, c and k_2 exist since the Lebesgue Stieltjes integral with respect to the variable l is finite. The operator T maps G onto itself, as clearly seen all the functions $T(x(t))$ with $x \in G$ are equicontinuous since they all have uniformly bounded derivatives. Now, we shall find an estimate of the modulus of continuity of the functions. $DT(x)(t)$ for $t \in [t_0, t_1]$

$$|DT(x)(t) - DT(x)(s)| \le m(t)\|x(t)\| - m(s)\|x(s)\|$$
$$+ \left| \int_{-h}^{0} d_s H(t_1, s, z(t))u(l+s) - d_s H(l, s, z(t))u(l+s) \right| \tag{3.7}$$
$$+ |f(t, x(t), \dot{x}(t), u(t)) - f(l, x(l), \dot{x}(l), u(l))|$$

The first two terms of the right-hand side of inequality (3.7) can be estimated as $\beta_0(t-l)$ where β_0 is a non-negative function and $\lim_{h\to 0} \beta_0(h) = 0$

In the same manner, we find that the term on the right of (3.7) can be estimated from condition (3.2)(d) as

$k|x(t) - x(l)| + \beta_1|t - 1|$

Letting $\beta = \beta_0 + \beta_1$

we finally obtain

$|DT(x)(t) - DT(x)(l)| \le k|x(t) - x(l)| + \beta|(t-1)|$

Hence, we conclude that for any set $Q \in G$, $\mu(TQ) < k\mu(Q)$

Consequently, by the Darbo's fixed point theorem, the operator T has at least one fixed point, therefore there exists a function.

$x^* \in C'([-h,0], E^n)$, such that

$$x(t) = x^*(t) = T(x^*)(t) \tag{3.8}$$

Differentiating with respect to t, we see that $x(t)$ as given by (3.8) is a solution to system (2.1); for the control $u(t)$ given by (3.4). The control $u(t)$ steers the system (2.1) from the initial complete state $y(t_0)$ to the desired vector $x_1 \in E^n$ on the interval $[t_0, t_1]$ and since $y(t_0)$ and x_1 have been chosen arbitrarily, then by definition 2.2, the system (2 2) is relatively controllable on $[t_0, t_1]$

CONCLUSION

In this study, we have examined sufficient conditions for the relative controllability of nonlinear systems with distributed delays with implicit derivative in the linear base and perturbation function in the affirmative by the use of Darbo's fixed point theorem

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