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ON THE PARAMETER ESTIMATION OF FIRST ORDER IMA MODEL CORRUPTED WITH WHITE NOISE

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ABSTRACT

In this paper, we showed how the autocovariance functions can be used to estimate the true parameters of IMA(1) models corrupted with white noise We performed simulation studies to demonstrate our findings. The simulation studies showed that under the presence of errors in not more than 30% of total data points, our method will very closely estimate the true parameter of the process

KEYWORDS: ARMA, Variance, Autocovariance Function, Parameter Estimation

1.0 INTRODUCTION

Consider the first order moving average model

$$w_t = a_t + \theta\, a_{t-1} \qquad 1.1$$

where,

w_t is the observed process

a_t is a white noise process. ie, it is distributed with zero mean and constant variance σ_a^2

θ is a weight parameter

Equation 1 1 is said to be invertible if the expansion $(1-\theta)^{-1}$ converges in square mean and this is the case where, $|\theta| < 1$, Priestley(1971).

Our interest is the case where equation 1.1 is not invertible but would be through the transformation $w_t - w_{t-1}$ .Doing this will result to the first order integrated moving average IMA(1) model of the form (Box and Jenkins1976)

$$(1-B)w_t = a_t + (\theta - 1)a_{t-1} \qquad 1.2$$

where B is a backward operator ($Ba_t = a_{t-1}$).

Further more, we postulate the case where w_t is actually a corruption of the true process z_t through $w_t = z_t - e_t$ where e_t is an error component introduced by faulty measurement or observation processes and is a white noise process.(Box and Jenkins 1976)

Substituting into equation1.2 we have

$$(1-B)z_t = a_t + (\theta - 1)a_{t-1} + e_t - e_{t-1} \qquad 1.3$$

where e_t is a white noise process uncorrelated with a_t .

Our interest is to estimate z_t through $w_t = z_t - e_t$.

We note the following known facts ,see Hamilton (1994) for example, for white noise processes

$$E(a_t a_{t-i}) = \begin{cases} \sigma_a^2 & for \quad i = 0 \\ 0 & for \quad i \neq 0 \end{cases}$$

$$E(e_t e_{t-i}) = \begin{cases} \sigma_e^2 & for \quad i = 0 \\ 0 & for \quad i = 0 \end{cases} \qquad 1.4$$

$$E(U_t U_{t-k}) = \begin{cases} \sigma_U^2 & for \quad k = 0 \\ 0 & for \quad k \neq 0 \end{cases}$$

$$E(e_{t-i} a_{t-i}) = 0 \qquad e_t, a_t \ are\ uncorrelated$$

where U_t is also a white noise process.

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Moran (1971) has shown that if the ratio $\lambda = \frac{\sigma_a^2}{\sigma_e^2}$ is known, then the maximum likelihood estimates for the parameter set can be found. The maximum likelihood estimates for the case where both σ_a^2 *and* σ_e^2 are known (the so called "over verification case") are estimated by Barnet (1967) by directly solving the likelihood equation. Chan and Mak (1979) obtained the maximum likelihood estimates for the case where both σ_a^2 *and* σ_e^2 are unknown and where the observations are replicated.
Our interest is to use autocovariance function to estimate the parameter values of the real series.

2.0 AUTOCOVARIANCE FUNCTIONS AND TRUE PARAMETER ESTIMATE

We can write equation1.3 as

$$x_t = a_t + (\theta-1)a_{t-1} + e_t - e_{t-1} \qquad 1.5$$

where $x_t=(1-B)z_t$.

From equation 1.5 we can set

$$x_t x_t = \{ a_t + (\theta-1)a_{t-1} + e_t - e_{t-1} \}\{ a_t + (\theta-1)a_{t-1} + e_t - e_{t-1} \}$$

Expanding and using the set of equations in 1.4 to take expectations we have

$$V_0 = (1+(\theta-1)^2)\sigma_a^2 + 2\sigma_e^2 \qquad 1.6$$

And

$$x_t x_{t-1} = \{ a_t + (\theta-1)a_{t-1} + e_t - e_{t-1} \}\{ a_{t-1} + (\theta-1)a_{t-2} + e_{t-1} - e_{t-2} \}$$

Expanding and using the set of equations in 1 4 to take expectations we have

$$V_1 = (\theta-1)\sigma_a^2 - \sigma_e^2 \qquad 1.7$$

Solving equations 1.6and 1.7 for σ_e^2 and σ_a^2 we have

$$\sigma_e^2 = \frac{(\theta-1)V_0 - \{ 1+(\theta-1)^2 \}V_1}{\theta^2} \qquad 1.8$$

and

$$\sigma_a^2 = \frac{V_0 + 2V_1}{\theta^2} \qquad 1.9$$

We obtain the parameters of IMA (1) model which represents the observed process by re-writing (1.1) as (Box and Jenkins (1976)

$$x_t = U_t + \alpha U_{t-1} \qquad 1\ 10$$

or in the IMA form of equation 1.2 as

$$x_t = U_t + (\alpha-1)U_{t-1} \qquad 1\ 11a$$

where

$U_t = a_t - e_t$ is a white noise process.

Consider

$$U_t U_t = (a_t - e_t)(a_t - e_t) \qquad 1.11b$$
$$= a_t a_t - 2a_t e_t + e_t e_t$$

Taking expectations on equation 1 11b and using the set of equations in 1.4, we have

$$\sigma_t^2 = \sigma_a^2 + \sigma_e^2$$

We note that the process 1.5 and 1.10 are equivalent with the same variance autocovariances.

We can obtain the parameters σ_U^2 and α using method of maximum likelihood.

The process (1.11a) has the variance

$$x_t x_t = \{ U_t + (\alpha-1)U_{t-1} \}\{ U_t + (\alpha-1)U_{t-1} \} \qquad 1\ 12$$

Expanding and using the set of equations in 1 4 to take expectations we have

$$V_0 = (1+(\alpha-1)^2)\sigma_u^2$$

$$V_0 = \left(1+(\alpha-1)^2\right)\left(\sigma_a^2+\sigma_e^2\right)$$

$$= \left(1+(\alpha-1)^2\right)\left[\frac{(\theta-1)V_0 \quad \{1+(\theta-1)^2\}V_1}{\theta^2}+\frac{V_0+2V_1}{\theta^2}\right]$$

$$= \left(1+(\alpha-1)^2\right)\left[\frac{\theta V_0+\{1-(\theta-1)^2\}V_1}{\theta^2}\right]$$

this gives

$$\theta^2 V_0 = (1+(\alpha-1)^2)(\theta V_0 + (1-(\theta-1)^2)V_1$$

or

$$f(\theta) = \theta^2 V_0 - \{1+(\alpha-1)^2\}\{\theta V_0 + [1-(\theta-1)^2]V_1\} \qquad 1.13$$

Our objective is to estimate the parameter θ. However, equation 1.13 is non-linear and can be solved by the Newton-Raphson process. In this case, the θ_{i+1} solution is obtained from the i^{th} approximation according to

$$\theta_{i+1} = \theta_i - \frac{\theta_i^2 V_0 - \{1+(\alpha-1)^2\}\{\theta_i V_0 + [1-(\theta_i-1)^2]V_1\}}{2\theta_i V_0 - \{1+(\alpha-1)^2\}\{V_0+2(1-\theta_i)V_1\}} \qquad 1\ 14$$

The denominator is the derivative of equation 1.13 with respect to θ.

We start the iteration with $\theta = \alpha$. At the point of convergence, we expect the value of θ to be such that $f(\theta) = 0$

3.0 SIMULATION STUDY

The objective of the simulation study is to rate the performances or responses of the iteration formula in 1.14 to various levels of errors. The technique is to first simulate an IMA model that follows the conditional mean equation

$$x_t = a_t + 1.38a_{t-1} \qquad 1\ 15$$

chosen to ensure non invertibility.

We difference equation1.15 to get the IMA form, which is invertible.

$$W_t = a_t + 0.38a_{t-1} \qquad 1.16$$

1000 data points where simulated to follow equation 1.15 and then differenced to follow equation 1.16. We then follow Muller and Yohai (2002) to introduce errors at equally spaced interval by defining a series with additive errors as

$$x_t = \begin{cases} y_t + b_t & if \quad t = t_i \\ y_t & elsewhere \end{cases} \qquad 1.17$$

where $t = t_i$, i=1,2...,are the times when errors where observed. The quantity, b_t is a sequence of normal random variables.

We looked at the cases where (i) no error was introduced, (ii) errors are introduced in 100 data points representing 10% of 1000 data points, (iii)errors are introduced in 200 data points representing 20% of 1000 data points ,(iv) errors are introduced in 300 data points representing 30%of 1000 data points, (v) errors are introduced in 400 data points representing 40%of 1000 data points ,(vi) errors are introduced in 500 data points representing 50%of 1000 data points.

The main idea is to determine the level of error that can be handled by the iteration formula in equation 1.14

3.1 RESULTS

Table1 shows the estimate of θ (which is α) for the simulated series with various percentages of errors. We note that the estimates increasingly become unreliable as percentage errors increases. For example, the estimate of θ (which is α) for an error of 50 percent of total data points, has the value 1.4969. This result is far away from the true value of 1.38 as seen from the conditional mean equation 1.15. The Mcleod and Sales(1983) algorithm was used for the maximum likelihood estimation.

Table 1: Maximum Likelihood Estimates Of The Simulated Model With Various Levels Of Percentage Error In The Number Of Data Points.

Percentage error in the number of data points	V_0	V_1	α
0	1.4305	0.4750	1.38
10	1.4446	0.4706	1.3872
20	1.4584	0.4664	1.3939
30	1.4945	0.4503	1.4094
40	1.5686	0.4250	1.4404
50	1.6849	0.4378	1.4969

Note: α is estimate of true parameter θ

3.2 TRUE PARAMETER ESTIMATE θ

We now use the iterative formula 1.14 to estimate the true parameter value θ by substituting into the formula the values of V_0 and V_1 in Table1 and using the value of α to start the iteration.. We demonstrate the result for errors of 20 and 50 percent of data points. The general result is presented in Table2.We note from table 2 that while the iteration produces good estimates up to error 30% of total data point, it estimates are poor for the case of errors of 40% and 50% of data points.

Iteration for 20% errors		Iteration for 50% errors	
Iteration Number	Estimate	Iteration Number	Estimate
0	1.3939	0	1.4969
1	1.3831	1	1.4340
2	1.3831	2	1.4312
		3	1.4312

Table 2: Estimate of the true parameter of the real series from result in table 1

Percentage error in the number of data points	θ
0	1.38
10	1.3816
20	1.3831
30	1.3842
40	1.3910
50	1.4312

CONCLUSION

We have shown that autocorrelation functions can be used to estimate the true parameter of an IMA(1)model white noise errors if the errors are not present in more than 30% of data points..

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