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MATHEMATICAL MODELLING AND ITS IMPACT ON OPEN CHANNEL FLOW

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ABSTRACT

Mathematical model for dredging (excavating) an open channel, namely, a river has been developed using the conditions for best hydraulic performances for the channel. Applying the model to a numerical example we determine new dimensions for the new open channel for two-channel sections, viz: the trapezoidal and rectangular sections. Comparison of the two channel sections based on our model test reveals that for each channel the new discharge and new mean velocity are greater than the original one if the cross sectional area, bed slope and surface roughness remain unchanged for each channel, and if the side slopes are also stable with respect to trapezoidal channel. The new depth and the new hydraulic mean depth are greater than the original one. The rectangular section is hydraulically and economically better than the trapezoidal section. A combination of our model with Darcy's formula can provide an alternative method of comparing the hydraulic performances of the two channel sections.

KEYWORDS: Open channel, hydraulic performance, dredging, mathematical model.

INTRODUCTION

An open channel is a duct through which a liquid flows with a free surface; for example, canals, rivers and pipes which are not running full (Chow 1959). The pressure on the free surface is usually atmospheric. The flow is not due to pressure differences along the channel but is caused by differences in the potential energy head due to the slope of the channel (Chow 1959).

Various investigations have been done in open channel flows. For instance, in studying flow in a channel with a slot in the bed, Nasser et al, (1980) tried to provide an insight into some aspects of spatially varied open channel flow. Other investigators include, notably, Bradley and Peterka (1957), Gill (1980), Repogle (1962), Rouse (1936), White, Charlton and Ramsay (1972) and Yallin (1971).

In this work the flow is assumed to be uniform and steady. Mathematical model governing the dredging of trapezoidal and rectangular channels is developed. The two channels are compared hydraulically and economically. Finally, an alternative method for predicting hydraulic performances of the two channels based on the percentage decrease in head loss for each channel has been developed.

CONDITIONS OF HYDRAULIC PERFORMANCE

(a) Trapezoidal section of a channel

Consider a trapezoidal section of a channel shown in Figure 1 with side slopes of 1 vertical to k horizontal. Let h be the depth of flow, b the bottom width and B the surface width.

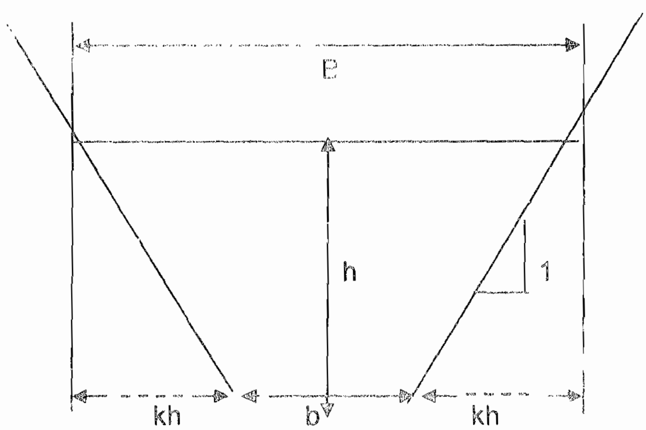

Figure 1: Trapezoidal section of an open channel

The cross sectional area of the trapezoidal channel is:

$$A_T = h\,(b + kh) \qquad (1)$$

where the subscript T refers to the trapezoidal channel.

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The wetted perimeter is

$$P = b + 2h\,(1 + k^2)^{1/2} \qquad (2)$$

Therefore the hydraulic mean depth is (Chow 1959)

$$M = \frac{A_T}{P} = \frac{h\,(b + kh)}{b+2h\,(1 + k^2)^{1/2}} \qquad (3)$$

From (3) we find

$$b = \frac{A_T}{h} - kh \qquad (4)$$

Substituting (4) into (3) we have

$$M = \frac{A_T h}{A_T + \alpha\, h^2} \qquad (5)$$

where

$$\alpha = 2\,(1 + k^2)^{1/2} - k \qquad (6)$$

For effective hydraulic performance the hydraulic mean depth M must be maximum for a given value of A_T (Yallin 1971). This maximum occurs when $\frac{dM}{dh} = o$. Thus

$$\frac{dM}{dh} = \frac{A_T\,(A_T + \alpha\, h^2) - 2\,\alpha\, h\,(A_T h)}{(A_T + \alpha\, h^2)^2} = 0$$

Hence

$$A_T = \alpha\, h^2 \qquad (7)$$

Substituting (7) into (5) for conditions of efficient hydraulic performance we find

$$M_{efficient} = \frac{h}{2} \qquad (8)$$

for trapezoidal channel.

Combining (4) , (6) and (7) we obtain another condition

$$\frac{b}{h} = 2(1 + k^2)^{1/2} - 2k \qquad (9)$$

which determines the relationship of bottom width to depth of flow.

(b) Rectangular section

The cross sectional area is (see Figure 2)

$$A_R = hb \qquad (10)$$

Here the subscript R denotes the rectangular channel

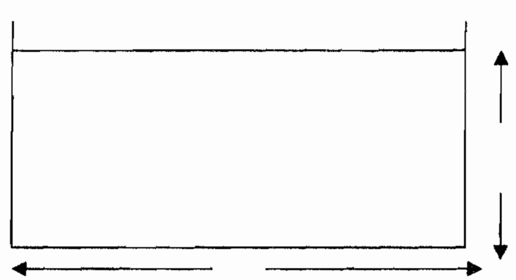

Figure 2: Rectangular section of an open channel

The wetted perimeter is

$$P = b + 2h \qquad (11)$$

while the hydraulic mean depth is

$$M = \frac{A_R}{P} = \frac{hb}{b+2h} \qquad (12)$$

From (10)

$$b = \frac{A_R}{h} \tag{13}$$

so that (11) becomes

$$P = \frac{A_R}{h} + 2h \tag{14}$$

For hydraulic effectiveness the wetted perimeter must be a minimum (Yallin 1971). Thus differentiating (14) with respect to h and setting it equal to zero we find

$$\frac{dP}{dh} = - A_R h^{-2} + 2 = 0$$

This yields

$$A_R = 2h^2 \tag{15}$$

Substituting (15) and (13) into (12) we have

$$M_{efficient} = \frac{b}{4} \tag{16}$$

as a condition of hydraulic performance for the rectangular channel.

DEVELOPMENT OF MATHEMATICAL MODEL FOR DREDGING AN OPEN CHANNEL

Throughout, the two systems in dredging an open channel shall be denoted by the symbols 0 and N where
System 0 = original open channel (i.e. open channel before dredging).
System N = new open channel (i.e. open channel after dredging).

Mathematical model for trapezoidal channel

(i) Determination of new depth h_N

From (9)

$$b_N = h_N [2(1+k^2)^{½} - 2k] \tag{17}$$

Since the cross sectional area is constant for both original and new channel, we find

$$h_N (b_N + kh_N) = A_T = h_0(b_0 + kh_0) \tag{18}$$

Combining (17) and (18) and simplifying we obtain

$$h_N = \left[\frac{A_T}{2 (1+k^2)^{½} - k}\right]^{½} \tag{19}$$

(ii) Determination of new width b_N

Substituting (19) into (17) we find

$$b_N = \left[\frac{A_T}{2 (1+k^2)^{½} - k}\right]^{½} \left[2(1+k^2)^{½} - 2k\right] \tag{20}$$

(iii) Determination of new hydraulic mean depth M_N

From (8)

$$M_N = \frac{h_N}{2} \tag{21}$$

(iv) Determination of new discharge Q_N

From Mannings formular (Chow 1959)

$$u = \frac{1}{n} (M)^{2/3} S_o^{½} \tag{22}$$

where u = mean velocity, n = Manning's roughness factor, S_o = bed slope.

From the relation Q = Au we obtain

$$Q_N = \frac{1}{n} A_T (M_N)^{2/3} S_o^{1/2} \tag{23}$$

(iv) Determination of new mean velocity u_N
Using (22) we find

$$u_N = \frac{1}{n} (M_N)^{2/3} S_o^{1/2} \tag{24}$$

or

$$u_N = \frac{Q_N}{A_T} \tag{25}$$

Mathematical model for rectangular channel

(vi) Determination of h_N

From (15)

$$h_N = \left(\frac{A_R}{2}\right)^{1/2} \tag{26}$$

(vii) Determination of b_N
For constancy in the cross sectional area, we have

$$b_N h_N = A_R = b_0 h_0 \tag{27}$$

Substituting (26) into the first half of (27) we obtain

$$b_N = (2A_R)^{1/2} \tag{28}$$

(viii) Determination of M_N
From (16)

$$M_N = \frac{b_N}{4} \tag{29}$$

(ix) Determination of Q_N
Again, from Manning's formular (22) we obtain

$$Q_N = \frac{1}{n} A_R (M_N)^{2/3} S_o^{1/2} \tag{30}$$

Determination of u_N
This is given by

$$u_N = \frac{Q_N}{A_R} \tag{31}$$

The expressions (19), (20), (21), (23), (24) or (25) constitute the mathematical model for dredging trapezoidal channel while the expressions (26), (28), (29), (30) and (31) constitute the model in respect of the rectangular channel.

ALTERNATIVE METHOD FOR COMPARING HYDRAULIC PERFORMANCE

From Darcy's formular (Chow 1959) the head loss h_f due to friction in an open channel is

$$h_f = \frac{fL}{M} \cdot \frac{u^2}{2g} \tag{32}$$

Thus

Head loss in the original channel: $(h_f)_0 = \frac{f L_0}{M_0} \cdot \frac{(u_0)^2}{2g}$ (33)

Head loss in the new channel: $(h_f)_N = \frac{f L_N}{M_N} \cdot \frac{(u_N)^2}{2g}$ (34)

Decrease in head loss in the channel due to dredging is

$$(h_f)_0 - (h_f)_N = \left(\frac{f L_0}{M_0} \cdot \frac{(u_0)^2}{2g}\right) - \left(\frac{f L_N}{M_N} \cdot \frac{(u_N)^2}{2g}\right) \quad (35)$$

The relation (35) is an alternative model for comparing hydraulic performances of the two-channel sections. Here L_0, L_N are the characteristic lengths of the original channel and new channel respectively, while f is the friction factor corresponding to Manning's n. u_0, u_N, M_0, M_N have their usual meanings for the two channels.

APPLICATION OF THE MODEL TO NUMERICAL EXAMPLE

Consider, for example, a channel with bed slope 1 in 500, bottom width 20m and conveying water at a depth 5m, Manning's coefficient n is 0.012. Using the model we wish to determine after dredging the discharge of the original channel, the dimensions of a channel to give the maximum discharge, the new discharge, the new mean velocity and the percentage decrease in head loss in
Trapezoidal section of sides 1 vertical to 2 horizontal
Rectangular section

Solution:

(a) Trapezoidal section:
For the original channel:
$B_0 = 20m$, $h_0 = 5m$, $k = 2$, $n = 0.012$, $S_0 = 1/500$, $A_T = 150m^2$,
$P_0 = 42.36m$, $M_0 = 3.54m$, $Q_0 = 1296.88m^3/s$, $u_0 = 8.64m/s$

The dimensions h_N and b_N of the new (excavated) trapezoidal channel are obtained respectively by substituting the appropriate data above in the model expressions (19) and (20). Thus

$$h_N = \sqrt{\frac{150}{2\sqrt{5} - 2}} = 7.78m$$

$$b_N = \sqrt{\frac{150}{2\sqrt{5} - 2}} \times (2\sqrt{5} - 4) = 3.65m$$

Other parameters of the new channel, namely, M_N, Q_N, u_N are determined respectively from the model expressions (21), (23), (24) or (25) via

appropriate substitution. We find

$$M_N = \frac{7.78}{2} = 3.89m$$

$$Q_N = (1/0.012)\,(150)\,(3.89)^{2/3}\,(1/500)^{1/2} = 1381.39 \text{ m 3/s}$$

$$u_N = \frac{1381.39}{150} = 9.20m/s$$

Table 1: Result for trapezoidal channel

	Trapezoidal channel	
	Original channel	New Channel
Side slope	1 vertical to 2 horizontal	1 vertical to 2 horizontal
Bed slope S_o	1/500	1/500
Mannings' n	0.012	0.012
Area of cross section A_T	$150m^2$	$150m^2$
Width b	20m	3.65m
Depth h	5m	7.78m
Wetted perimeter P	42.36m	38.41m
Hydraulic mean depth M	3.54m	3.89m
Discharge Q	$1296.88m^3/s$	$1381.39m^3/s$
Mean velocity u	8.64m/s	9.20m/s
Head loss h_f	0.2579m	0.0485

(b) Rectangular section

Here

$b_0 = 20m$, $h_0 = 5m$, $n = 0.012$, $S_0 = 1/500$, $A_R = 100m^2$,
$P_0 = 30m$, $M_0 = 3.33m$, $Q_0 = 830.33m^3/s$, $u_0 = 8.30m/s$

for the original channel. Similar substitution of the appropriate original data of the rectangular channel into the model expressions (26) and (28) yields receptively the following dimensions h_N and b_N. That is

$$h_N = \sqrt{\frac{100}{2}} = 7.07m$$

$$b_N = \sqrt{(2 \times 100)} = 14.14m$$

The corresponding parameters M_N, Q_N, u_N of the rectangular section are determined respectively through appropriate substitution in the expressions (29), (30) and (31). Thus

$$M_N = \frac{14.14}{4} = 3.53m$$

$$Q_N = (1/0.012)\,(100)\,(3.53)^{2/3}\,(1/500)^{1/2}$$

$$= 863.24m^3/s$$

$$u_N = \frac{863.24}{100} = 8.63m/s$$

From the alternative method (35) we obtain

(a) Decrease in head loss in trapezoidal channel due to dredging:

$$(h_f)_o - (h_f)_N = \left[\frac{0.012 \times 20}{3.54}\right]\left[\frac{(8.64)^2}{2 \times 9.81}\right] - \left[\frac{0.012 \times 3.65}{3.89}\right]\left[\frac{(9.20)^2}{2 \times 9.81}\right]$$

$$= 0.2094$$

∴ Percentage decrease in head loss = 81.19%

(b) Decrease in head loss in rectangular channel due to dredging :

$$(h_f)_o - (h_f)_N = \left[\frac{0.012 \times 20}{3.33}\right]\left[\frac{(8.30)^2}{2 \times 9.81}\right] - \left[\frac{0.012 \times 14.14}{3.53}\right]\left[\frac{(8.63)^2}{2 \times 9.81}\right]$$

$$= 0.0706$$

∴ Percentage decrease in head loss = 27.89%

Table 2: Result for rectangular channel

	Rectangular Channel	
	Original channel	New Channel
Bed slope S_0	1/500	1/500
Manning's n	0.012	0.012
Area of cross section A_R	$100m^2$	$100m^2$
Width b	20m	14.14m
Depth h	5m	7.07m
Wetted perimeter P	30m	28.28m
Hydraulic mean depth M	3.33m	3.53m
Discharge Q	$830.33m^3/s$	$863.24m^3/s$
Mean velocity u	8.30m/s	8.63m/s
Head loss h_f	0.2530	0.1824

CONCLUSION

The new dimensions (b_N, h_N) for the trapezoidal and rectangular sections based on the model are shown respectively in Tables 1 and 2 The tables also show that for each channel the new discharge, new mean velocity, new depth and new hydraulic mean depth are greater than the original ones. Though the discharge in respect of the trapezoidal section is greater than that of the rectangular section, hydraulically, the rectangular section is more effective than the trapezoidal section because the former has the minimum perimeter for a particular cross sectional area. Using the results of our analysis in (35) the percentage decrease in head loss due to friction in each channel is determined. Our result shows that this percentage is smaller in the rectangular channel than in the trapezoidal one. This further shows that hydraulically the rectangular section is more effective than the trapezoidal section. Furthermore, the rectangular section is economically better than the trapezoidal section in the sense that since the wetted perimeter of the former is a minimum it requires minimum excavation and lining costs.

It is noted generally that for a fixed cross sectional area, the depth of the channel will increase but the width will decrease (Tables 1 and 2) with consequent increase in the capacity of the channel to carry water for navigation.

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