GLOBAL JOURNAL OF MATHEMATICAL SCIENNCES VOL. 5, NO. 2, 2006: 147-152
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# AGE-DEPENDENT MARKOV MODEL OF MANPOWER SYSTEM

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(Received 5 July, 2005; Revision Accepted 16 August, 2005)

ABSTRACT

We consider a Markov Model based on a Transition Probability Matrix (TPM) that would be used in the projection of the structure of a manpower system in which employees are classified according to age. The TPM is assumed to be stationary. The method of maximum likelihood is used in the estimation of the transition probabilities. The model is applied to the academic manpower system of the University of Nigeria, Nsukka.

**KEY WORDS**: Markov model ,Manpower system, Maximum likelihood, Transition probabilities

## INTRODUCTION

Manpower system is defined as any identifiable group of people working with a common end in view, Bartholomew (1971). It can be described in terms of stocks of flows. The stocks represent the number of people employed in the system at a particular time and for many reasons may be broken down into a set of homogenous classes based on some characteristics of their members such as age, length of service, grade etc. The stocks are subject to a number of flows which include: movement into the system (recruitment), movement within the system between classes in the form of promotion and interval transfers and movement out of the system known as wastage. Usually the recruitment and promotion flows are under the control of management. However, the wastage flow is stochastic in nature and this calls for statistical expertise, Bartholomew et al (1991).

Within the total wastage, there are two important subdivisions, which must be carefully distinguished in practice. The division is between voluntary and involuntary wastage. Voluntary wastage arises whenever an individual leaves the system of his own choice usually to take another job, this is a highly variable phenomenon since it is a result of many unpredictable individual decisions. Involuntary wastage, on the other hand, covers all losses for reasons beyond the control of the individual such as death, ill health, redundancy and retirement.

The manpower planner therefore requires to predict such flows and use the current stock to estimate the stocks at some future date. Though a manpower planner may also be required to determine the optimal pathway to accomplish some given management objectives, it may be desirable to control the flows so as to produce a desired structure. Such control problems have two aspects: attainability and maintainability where attainability is concerned with the feasibility of attaining a desired structure and maintainability is a process of maintaining the existing structure. Bartholomew, (1982).

The earliest works in modern manpower modelling are to be found in Seal 1945, Vajda (1947, 1948) arising out of their work during the Second World War. Though the full potential of their models could not be exploited due to inadequate computing resources. Young and Almond (1961) were the first to propose the transition model where the Markov model was applied in forecasting the future development of an expanding firm with stable recruitment.
In this work we intend to:

(i) Describe a suitable age-dependent Markovian model based on a transition probability matrix (TPM) that can be used in the projection of the structure of a manpower system.
(ii) Apply the model to the academic manpower system of University of Nigeria with a view to obtaining among other things:
(a) The expected further duration of an academic staff of university of Nigeria in the various age classes.
(b) The limiting age structure of the academic manpower system of the university of Nigeria, Nsukka.

### Notations and Terminology

| | |
|---|---|
| K | -number of age categories |
| $n_i(T)$ | -stock in category i at T time period |
| $n(T)$ | -{ $n_1(T)$, $n_2(T)$ ..., $n_k(T)$} the stock vector |
| $n_{ij}(T-1)$ | -the number of individuals moving from category i to j in the unit interval (T-1, T). |
| $n_{i\,k+1}(T-1)$ | -the wastage flow from category i in the interval (T-1, T) |
| $n_{oi}(T)$ | -the recruitment flow into category i in the interval (T-1, T) |
| $N(T)$ | -the total size of the system at time T |
| $P_{oi}(T)$ | -probability of recruitment into category i in (T) |
| $P_{i\,k+1}(T)$ | -the probability that an individual in age class i at (T-1) is out of the system in (T) |
| $P_{ij}(T-1)$ - | probability of transition from age category i into j in the interval (T, T-1). |
| $P = (P_{ij})$ | - the transition probability matrix |

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The system under consideration is made of stock and flows. Stock refers to the number of employees at the various age categories at any time period while flow refers to the movement from one age class to the next, movement out of the system (wastage) and movement into the system (recruitment). Recruitment includes fresh appointment, transfer of service into the system and can be made into any of the K age categories at any time period. Let the total number of recruits at time T be denoted by R(T) and these recruits be allocated to the jth age classes with probability $P_{0j}$ such that

$$\sum_{j=i}^{k} P_{oj} = 1.$$

$$\sum_{j=1}^{k} P_{ij} + P_{i\,k+1} = 1$$

The basic Markov assumption is that the flows are governed by transition probabilities and that the classes are homogeneous and independent with respect to these probabilities

Wastage in the system is the process of leaving the system for whatever reason . It should be noted that since each person either stay where he is, or move to the next class or leave, the row sum of the transition must be equal to 1

**The Model when R (T) is fixed**

Using notations above, we can represent stock in category j mathematically as follows

$$n_j(T) = \sum_{i=1}^{k} n_{ij}(T-1) + n_{oj}(T) \text{-------------------} (1)$$

$i = 1,2,\ldots,k$ and $j = 1,2,\ldots,k$

Since the flow is random, we can talk of their expectation. Hence

$$\bar{n}_j(T) = \sum_{i=1}^{k} \bar{n}_{ij}(T-1) + \bar{n}_{oj}(T) \text{-------------} (2)$$

where $\bar{n}$ denotes an expectation.

Given the expected number at the start of the period $n_i$ (T-1), and the total recruitment R (T) it follows that expected flows are

$$\left.\begin{aligned} \bar{n}_{oj}(T) &= R(T)P_{oj} \\ \bar{n}_{ij}(T-1) &= n_i(T-1)P_{ij} \end{aligned}\right\} \ldots\ldots\ldots\ldots (3)$$

Hence, substituting equation (3) in (2) we have

$$n_j(T) = \sum_i n_i(T-1)P_{ij} + R(T)P_{oj} \ldots\ldots\ldots\ldots (4)$$

or in matrix notation

$$\bar{n}(T) = \bar{n}(T-1)P + R(T)\underline{r} \ldots\ldots\ldots\ldots (5)$$

where $\underline{r} = P_{0j}$

which is the basic stock prediction equation. If we are given the stock at any particular time, this equation enables us to predict what they will become one-step a head. By recursive use of the equation, we can predict into the longer-term future assuming the parameter values remain unchanged.

**The Model when N(T) is Fixed**

However, when stock size is fixed, the above model would change slightly. Here the number of recruits $R(T)r_j$ will now be a random variable composed of two parts. The first part consists of those recruited to fill any new vacancy arising from growth in the system and the second consists of those who replace leavers. The expected value of R(T) is thus

$$R(T) = N(T) - N(T-1) + \sum^{k} \overline{n}_i(T-1)W \text{ ........................ 6}$$

substituting (6) in (5) we have

$$\overline{n}(T) = \overline{n}(T-1)\{P + W'\underline{r}\} + M(T)\underline{r} \text{ ............................ (7)}$$

where $W = P_{i\,k+1}$ and $M(T) = N(T) - N(T-1)$

Each term in this equation can be identified in the following way:

n(T-1) P - represents normal internal movement from one age class to the next.
n (T-1) W'r - represents recruits who replace leavers
M(T)r - represents recruits filling new or created vacancies

We shall write

Q = P + W'r

The first term corresponding to a real direct flow from i to j while $W_i$ can be interpreted as a hypothetical indirect flow comprising the part of the wastage flow from i which goes back to j as recruitment.

ESTIMATION OF THE TRANSITION PROBABILITIES

It is easy to obtain point estimates of the transition probabilities from historical data by the method of maximum likelihood Bartholomew (1982). If $n_{ij}(T)$ is the observed number in i at T-1 who are in j at T and if $n_i(T)$ is the stock at the beginning of this interval then the estimator of $P_{ij}(T)$ is

$$\hat{P}_{ij}(T) = \frac{n_{ij}(T)}{n_i(T)} \text{ .......... .......... .......... .......... .......... ......( 8 )}$$

If the stock and flow data are available over several time intervals for which rates are assumed to be the same then

$$\hat{P}_{ij}(T) = \frac{\sum_T n_{ij}(T)}{\sum_T n_i(T)}$$

where summation is taken over the periods for which data are available.
The wastage and recruitment probabilities are estimated in the same way. i.e.

$$P_{oj}(T) = \frac{n_{oj}(T)}{\sum_j no_j(T)}$$

$$P_{ik+1}(T) = \frac{n_{ik+1}(T)}{n_i(T)}$$

**Average Length of Stay in the System**

In manpower planning, it is important to know in advance how long an employee is expected to stay in the system. This serves as a measure of the career prospect of an employee. The average length of stay of an employee in category i can be obtained thus

$$\mu = (I - P)^{-1}$$

This is also known as the expected further duration. The proof of the above is contained in Bartholomew (1982) and Bhat(1971)

**The Limiting Age Structure**

Although the Markov model is usually used to make projections in arithmetic terms, it is possible to gain valuable insight by mathematical analysis. One of the most useful things to know is the direction in which the structure is changing. This can be explored by investigating the limit of $\overline{n}$ (T) as $T \to \infty$.
Considering our model of constant R(T) = R say, then if a limiting structure exists, it must satisfy

$$\bar{n}(\infty) = \bar{n}(\infty) P + Rr$$

which can be solved for $\bar{n}(\infty)$ to have

$$\bar{n}(\infty) = Rr\,(I - P)^{-1}.$$

**Application**

In this section we intend to apply the model described in the last sections to the academic manpower data of the University of Nigeria as obtained from the Personnel Services Department of the University. Data in this respect refer to the number of employees in the various age classes,
the number of recruits in the various classes, the number of leavers in the various age classes. The age classes measured in years are as follows: - 21-25, 26-30, 31-35 ... 61-65.

For the purpose of convenience the age classes are represented serially from 1 to 9. The age class length of 5 was so chosen for the reasons of homogeneity and so that we can have a sizeable number of classes. The period under study is from 1998 to 2003. For the purpose of this work, leavers are those who have permanently withdrawn their services from the university and this excludes people on sabbatical leave, leave of absence and secondment.

**SUMMARY OF DATA:STOCK**

Table 1

| | Age Group | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Year | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Total |
| 1998 | 9 | 40 | 136 | 150 | 152 | 137 | 120 | 143 | 109 | 986 |
| 1999 | 7 | 40 | 134 | 150 | 142 | 146 | 140 | 139 | 108 | 1006 |
| 2000 | 3 | 36 | 76 | 116 | 133 | 211 | 210 | 141 | 114 | 1043 |
| 2001 | 4 | 34 | 78 | 114 | 137 | 211 | 209 | 136 | 106 | 1029 |
| 2002 | 6 | 39 | 140 | 168 | 159 | 133 | 235 | 136 | 79 | 996 |
| 2003 | 4 | 20 | 102 | 132 | 173 | 150 | 158 | 148 | 114 | 1001 |

Leavers

| | Age Group | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Year | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Total |
| 1998 | 1 | 5 | 3 | 6 | 4 | 2 | 8 | 3 | 11 | 43 |
| 1999 | 1 | 1 | 2 | 3 | 8 | 2 | 7 | 9 | 6 | 39 |
| 2000 | 0 | 4 | 5 | 8 | 4 | 3 | 7 | 3 | 6 | 40 |
| 2001 | 1 | 4 | 4 | 2 | 3 | 8 | 6 | 6 | 7 | 41 |
| 2002 | 3 | 2 | 6 | 3 | 4 | 3 | 10 | 5 | 2 | 38 |
| 2003 | 2 | 5 | 4 | 3 | 5 | 8 | 7 | 4 | 2 | 40 |

Recruits

| | Age Group | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Year | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Total |
| 1998 | 3 | 8 | 2 | 3 | 2 | 1 | 0 | 0 | 0 | 18 |
| 1999 | 3 | 5 | 5 | 2 | 4 | 1 | 0 | 0 | 0 | 20 |
| 2000 | 4 | 2 | 1 | 5 | 0 | 0 | 0 | 0 | 0 | 12 |
| 2001 | 3 | 5 | 3 | 1 | 2 | 1 | 1 | 0 | 0 | 16 |
| 2002 | 5 | 8 | 2 | 0 | 2 | 1 | 1 | 0 | 0 | 19 |
| 2003 | 1 | 6 | 4 | 1 | 1 | 1 | 1 | 0 | 0 | 15 |

Table 2. Transition Probability Matrix for the Entire Period $P_{ij}$

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Wastage |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.333 | 0.394 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.273 |
| 2 | 0 | 0.688 | 0.201 | 0 | 0 | 0 | 0 | 0 | 0 | 0.111 |
| 3 | 0 | 0 | 0.782 | 0.182 | 0 | 0 | 0 | 0 | 0 | 0.036 |
| 4 | 0 | 0 | 0 | 0.752 | 0.218 | 0 | 0 | 0 | 0 | 0.030 |
| 5 | 0 | 0 | 0 | 0 | 0.788 | 0.181 | 0 | 0 | 0 | 0.031 |
| 6 | 0 | 0 | 0 | 0 | 0 | 0.789 | 0.185 | 0 | 0 | 0.926 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0.778 | 0.176 | 0 | 0.046 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.774 | 0.190 | 0.036 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.946 | 0.054 |

Table 3. Transition Probabilities

| Prob. | 1998 | 1999 | 2000 | 2001 | 2002 | 2003 |
|---|---|---|---|---|---|---|
| $P_{11}$ | 0.667 | 0 | 0.667 | 0.5 | 0.333 | 0 |
| $P_{12}$ | 0.227 | 0.857 | 0.300 | 0.250 | 0.167 | 0.5 |
| $W_1$ | 0.111 | 0.143 | 0 | 0.250 | 0.500 | 0.5 |
| $P_{22}$ | 0.675 | 0.850 | 0.722 | 0.676 | 0.718 | 0.600 |
| $P_{23}$ | 0.200 | 0.125 | 0.167 | 0.206 | 0.231 | 0.150 |
| $W_2$ | 0.125 | 0.025 | 0.111 | 0.118 | 0.051 | 0.250 |
| $P_{33}$ | 0.787 | 0.828 | 0.810 | 0.821 | 0.736 | 0.715 |
| $P_{34}$ | 0.191 | 0.150 | 0.127 | 0.128 | 0.221 | 0.245 |
| $W_3$ | 0.022 | 0.022 | 0.063 | 0.051 | 0.043 | 0.040 |
| $P_{44}$ | 0.707 | 0.733 | 0.707 | 0.771 | 0.803 | 0.780 |
| $P_{45}$ | 0.253 | 0.247 | 0.224 | 0.211 | 0.179 | 0.197 |
| $W_4$ | 0.04 | 0.020 | 0.069 | 0.018 | 0.018 | 0.023 |
| $P_{55}$ | 0.803 | 0.719 | 0.767 | 0.760 | 0.793 | 0.867 |
| $P_{56}$ | 0.171 | 0.225 | 0.203 | 0.218 | 0.182 | 0.104 |
| $W_5$ | 0.026 | 0.056 | 0.030 | 0.022 | 0.025 | 0.029 |
| $P_{66}$ | 0.802 | 0.856 | 0.839 | 0.777 | 0.819 | 0.734 |
| $P_{67}$ | 0.183 | 0.130 | 0.147 | 0.185 | 0.158 | 0.213 |
| $W_6$ | 0.015 | 0.014 | 0.014 | 0.035 | 0.023 | 0.053 |
| $P_{77}$ | 0.741 | 0.700 | 0.795 | 0.803 | 0.785 | 0.715 |
| $P_{78}$ | 0.192 | 0.250 | 0.172 | 0.167 | 0.141 | 0.244 |
| $W_7$ | 0.067 | 0.050 | 0.033 | 0.030 | 0.074 | 0.044 |
| $P_{88}$ | 0.735 | 0.791 | 0.795 | 0.757 | 0.801 | 0.770 |
| $P_{89}$ | 0.244 | 0.144 | 0.186 | 0.220 | 0.162 | 0.203 |
| $W_8$ | 0.021 | 0.063 | 0.021 | 0.044 | 0.037 | 0.027 |
| $P_{99}$ | 0.899 | 0.944 | 0.947 | 0.934 | 0.975 | 0.982 |
| $W_9$ | 0.101 | 0.056 | 0.053 | 0.066 | 0.025 | 0.018 |

Table 4. Observed withdrawals in brackets and Expected Withdrawals to the Nearest Integer

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Total |
|---|---|---|---|---|---|---|---|---|---|---|
| 1998 | 2(1) | 4(5) | 5(2) | 5(6) | 5(4) | 4(2) | 6(8) | 5(3) | 6(11) | 42(43) |
| 1999 | 2(1) | 4(1) | 5(2) | 5(3) | 4(8) | 4(2) | 6(7) | 5(9) | 6(6) | 41(39) |
| 2000 | 1(0) | 4(4) | 3(5) | 3(8) | 4(4) | 5(3) | 10(7) | 5(3) | 6(6) | 41(40) |
| 2001 | 1(1) | 4(4) | 3(4) | 3(2) | 4(3) | 5(8) | 10(6) | 5(6) | 6(7) | 41(41) |
| 2002 | 2(3) | 4(2) | 6(6) | 5(3) | 4(4) | 3(3) | 6(10) | 5(3) | 4(2) | 39(36) |
| 2003 | 1(2) | 2(5) | 4(4) | 4(3) | 5(5) | 4(8) | 7(7) | 5(4) | 6(2) | 38(40) |

The Expected Further Duration

$$
(I-P)^{-1} = \begin{pmatrix}
1.499 & 1.893 & 1.746 & 1.281 & 1.317 & 1.130 & 0.942 & 0/733 & 2.58 \\
0 & 3.205 & 2.955 & 2.169 & 2.230 & 1.913 & 1.594 & 1.242 & 4.368 \\
0 & 0 & 4.587 & 3.366 & 3.462 & 2.97 & 2.475 & 1.927 & 6.781 \\
0 & 0 & 0 & 4.032 & 4.146 & 3.557 & 2.964 & 2.308 & 8.122 \\
0 & 0 & 0 & 0 & 4.717 & 4.046 & 3.371 & 2.626 & 9.239 \\
0 & 0 & 0 & 0 & 0 & 4.739 & 3.949 & 3.076 & 10.822 \\
0 & 0 & 0 & 0 & 0 & 0 & 4.505 & 3.508 & 12.343 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4.425 & 15.569 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 18.519
\end{pmatrix}
\quad
\begin{array}{c}
\text{Row Sums} \\
13.301 \\ 19.676 \\ 25.568 \\ 25.125 \\ 23.999 \\ 22.586 \\ 20.356 \\ 19.994 \\ 18.519
\end{array}
$$

**The Long run distribution of Staff**

Using the average number of recruits over the period which is R = 17, we have the following limiting age structure for the academic manpower of the University of Nigeria.

$\bar{n}(\infty)$ = 17[019 0.34 0.17 0.12 0.11 0.05 0.03 0.00 0.00] $[I-P]^{-1}$
= (4.842 24.639 35.976 34.627 44.427 42.14 37.415 29.138102.517)

Hence $\bar{n}(\infty)$ = 355

From the limiting structure, $\bar{n}(\infty)$, we have that the number of academic staff of University of Nigeria in the long run falls from the average of 1010 for the period under consideration to 355 academic staff, which shows that the system is a contracting manpower system, and management should initiate action toward control.

## CONCLUSION AND SUMMARY

We described an age-dependent Markovian model that can be used in projecting the structure of a manpower system in which employees are classified according to age, under the assumption of stationarity of the transition probability matrix and applied the model to the academic manpower data of the University of Nigeria for the period 1998 to 2003.

In the application, we estimated the transition probabilities using the method of maximum likelihood and using the model, we obtained the total expected withdrawals for the years under study to be 42 for 1998 41 for 1999, 2000 and 2001, 39 for 2002 and 38 for 2003 which compare well with the observed values of 43,39,40,41,36 and 40 for the years respectively. The result of the expected stock also compares well with the observed stock and this reveal the predictive potency of the model. To measure the career prospect of entrants, we estimated the expected length of stay in the system, the result showed that a recruit into age group 1 is expected to stay 13 years in the system and for age group 2 to 9 the expected length of stay are 20,26,25,24,23,20,20 and 19 respectively and the limiting age structure shows that the system is contracting.

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