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DERIVATION OF TRANSITION MATRIX FOR STAFF PROMOTION MARKOV MODEL USING PGF.

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ABSTRACT

In an establishment with n staff in a rank, s of them were promoted at Time $(i+1)$ with the probability $p_s(i+1,n)$. The Probability Generating Function (PGF) of this probability was used to derive a transition matrix that satisfies the assumptions of a staff promotion Markov chain model. The transition matrix of the staff promotion Markov model was applied to analyse academic staff promotion in a university. The theoretical results are interpreted in the light of the practical problem.

KEYWORDS: Markov Chains, transition matrix, promotion, Markov model.

1.0 INTRODUCTION

Recently, many models of Markov chains have been formulated and applied to numerous areas of human endeavour. For example, Davies et al (1975), and Schachtman and Hogue (1976) applied Markov models to solve some health sector problems. Others include the use of Markov chains to model the spread of epidemic diseases as can be seen in Gani (2003). The classical random allocation model arises quite naturally in the context of needle sharing among Intravenous Drug Users (IVDU), and was developed to model the growth of infective IVDUs by Gani (1991, 1993, 2002a, 2002b) and Gani and Yakowitz (1993). Here it is assumed that i hypodermic needles are used by IVDU who could be infected with a virus, such as hepatitis or HIV. These needles are randomly shared among n susceptible IVDU infecting $1 \le s \le min(i,n)$ of them.

Furthermore, applications of Markov Chains models in brandswitching market analysis and in analysis of a single conveyor system can be found in Dinkel et al (1978), Carey and Sherr (1974) and Gregory and Litton (1975). In formulating equipment replacement policy Markov models can be used to determine the proportion of each condition (state) of equipments in an organization. In this paper attention is paid to manpower planning which is an aspect of what is generally referred to as population mobility, Dinkel et al (1978). In Nigeria, different tribes move in different proportions to different geographical regions to spend a certain part of their lifetimes. For example, many Ibos especially before the Nigerian Civil War migrated in different proportions to other geographical regions of Nigeria. If the different tribes/geographical locations are considered as states of a transition matrix, then the proportion of each tribe residing in its geographical location on the long run can be determined. Grinold (1976), Valliant and Milkovich (1977), and Simmons (1971) are some work done on manpower planning using Markov models. The following section discusses the staff promotion Markov chain model.

2.0 STAFF PROMOTION MARKOV CHAINS MODEL

In any public or private establishment promotion to a higher grade is one of the ways of rewarding a hardworking staff for his dedication and experience while on the other hand demotion is one of the ways of reprimanding a lazy and undedicated staff in an establishment.

Assumptions of the Model

The assumptions of the model are stated thus:
A staff who is due for promotion can only move to the next higher level at the one step transition.
Demotion is not allowed
The stationarity property of Markov chains holds.
Each time period is one calendar year

Before appraisal meetings are held for staff promotion, every staff of each grade level is informed of the deadline time i $(i \ge 0)$ for submitting necessary appraisal forms and documents that prove that he is qualified for the promotion. During appraisal meeting at time $(i+1)$ every staff case is considered according to its merit. Let s staff be due for promotion at the time $(i+1)$, and let the number of staff at a certain cadre be n. It follows that at the end of the promotion meeting at time $(i+1)$, s staff are promoted, if at the time i deadline s staff were found qualified and the one additional name that came in was already among the counted qualified staff and the probability of those qualified is $\frac{s}{n}$. Or $(s-1)$ qualified with probability $\left(1-\frac{s-1}{n}\right)$ and the additional name that came in was found qualified.

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Let $P_s(i,n) = P$ (s qualified staff at time i/n staff were initially in the given cadre)

We have the following recursive equation for the probability:

$$P_s(i+1,n) = P_s(i,n)\frac{s}{n} + P_{s-1}(i,n)\left(1-\frac{(s-1)}{n}\right) \tag{1}$$

With $s = 1, 2, \ldots min(i+1,n)$ and

In general the recursive equation for the Markov model is given by:

$$P_s(i+1,n) = P_s(i,n)h_s + P_{s-1}(i,n)\, g_{s-1} \tag{2}$$

with $h_s + g_s = 1$, since $h_s = \frac{s}{n}$, $g_s = \left(1-\frac{s}{n}\right)$ and $g_{s-1} = \left(1-\frac{s-1}{n}\right)$

The various values of the probability h_s have also been considered. For example Rutherford (1954) has studied the case where $h_s = p + cs$

with $0 < c \leq \frac{(1-p)}{n}$ in which h_s may now be a simple linear function of s and p.

Recall (2)

$$p_s(i+1,n) = p_s(i,n)h_s + p_{s-1}(i,n)g_{s-1}$$

let $$h_s = p + cs \tag{3}$$

$$\Rightarrow g_s \quad 1 - h_s = 1 - p - cs$$

$$g_s \quad 1 \quad p \quad cs$$

$$\therefore \quad g_{s-1} = 1 - p - c(s-1)$$

$$= q - c(s-1) \tag{4}$$

where $1 - p = q$

Substituting (3) and (4) in (2) we have

$$p_s(i+1,n) = p_s(i,n)(p+cs) + p_{s-1}(i,n)\left[q - c(s-1)\right] \tag{5}$$

DERIVING THE MODEL TRANSITION MATRIX USING PGF

The simplest way to deal with this recursive equation (5) is through the probability generating function $(pgf)\ f_{in}(u)$ of the probabilities $p_s(i,u)$. i.e.

$$f_{i,n}(u) = \sum_{s=0}^{min(i,n)} p_s(i,n)\, u^s \tag{6}$$

and

$$f_{i+1,n}(u) = \sum_{s=0}^{min(i+1,n)} p_s(i+1,n)\, u^s \tag{7}$$

$\therefore$ We rewrite (5) in the form:

$$\sum_{s=0}^{min(i+1,n)} p_s(i+1,n)\, u^s = \sum_{s=0}^{min(i,n)} p_s(i,n)(p+cs)u^s + \sum_{s=0}^{min(i,n)} p_{s-1}(i,n)\left[q - c(s-1)\right]u^s$$

which reduces to the form:

$$\sum_{s=0}^{min(i+1,n)} p_s(i+1,n)\, u^s = p\sum_{s=0}^{min(i,n)} p_s(i,n)u^s + c\sum_{s=0}^{min(i,n)} p_s(i,n)su^s$$

$$+\ u\left[q\sum_{s=1}^{min(i,n)} p_{s-1}(i,n)u^{s-1} - c\sum_{s=1}^{min(i,n)} p_{s-1}(i,n)(s-1)u^{s-1}\right] \tag{8}$$

Substituting (6) and (7) in (8) we have:

$$f_{i+1,n}(u) = pf_{i,n}(u) + cu\frac{\partial f_{i,n}(u)}{\partial u} + uqf_{i,n}(u) - u^2c\frac{\partial f_{i,n}(u)}{\partial u} \tag{9}$$

Where $u\frac{\partial f_{i,n}(u)}{\partial u} = \sum_{s=0}^{min(i,n)} p_s(i,n)su^s$ and $u\frac{\partial f_{i,n}(u)}{\partial u} = \sum_{s=1}^{min(i,n)} p_{s-1}(i,n)(s-1)u^{s-1}$

(9) is further reduced to the form:

$$f_{i+1,n}(u) = (p+qu)f_{i,n}(u) + cu(1-u)\frac{\partial f_{i,n}(u)}{\partial u} \tag{10}$$

We now proceed to solve (10). We consider the first few $pgfs$ for $n \geq 3$. This is so because when say $i = 1, 2, and\ 3$ and considering the summation in (6) where $s \leq min(i, n)$ and by probability $s \leq n$ then $n \geq 3$. Let $f_{0,n}(u) = 1$

$$f_{1,n}(u) = p + qu$$

$$f_{2,n}(u) = (p+qu)^2 + cqu(1-u)$$

$$f_{3,n}(u) = (p+qu)^3 + cqu(1-u)(p+qu) + cu(1-u)[2q(p+qu) + cq(1-2u)]$$

We now seek a recursion equation for the coefficients of the various powers of u.
Let us write for $n \geq i$

$$f_{i,n}(u) = a_0(i) + a_1(i)u + a_2(i)u^2 + \ldots + a_i(i)u^i \tag{11}$$

We now substitute (11) in (10) to obtain:

$$f_{i+1,n}(u) = (p+qu)\left[a_0(i) + a_1(i)u + a_2(i)u^2 + \ldots + a_i(i)u^i\right] + cu(1-u)\left[a_1(i) + 2a_2(i)u + \ldots + ia_i(i)u^{i-1}\right]$$

$$\therefore\ f_{i+1,n}(u) = (p+qu)a_0(i) + \left[(p+c)u + (q-c)u^2\right]a_1(i) + \left[(p+2c)u^2 + (q-2c)u^3\right]a_2(i) + \ldots + \left[(p+ic)u^i + (q-ic)u^{i+1}\right]a_i(i) \tag{12}$$

Thus we put the coefficient of u in (12) in matrix form as follows:

$$A(i) = \begin{bmatrix} a_0(i) \\ a_1(i) \\ a_2(i) \\ \vdots \\ a_{i-1}(i) \\ a_i(i) \end{bmatrix} = \begin{bmatrix} p & & & & & \\ q & p+c & & & & \\ & q-c & p+2c & & & \\ & & & \ddots & & \\ & & & q-(i-2)c & p+(i-1)c & \\ & & & & q-(i-1)c & p+ic \end{bmatrix} \begin{bmatrix} a_0(i-1) \\ a_i(i-1) \\ a_2(i-1) \\ \vdots \\ a_0(i-1) \\ 0 \end{bmatrix} \tag{13}$$

In the system (13), the transpose of the square matrix is a transition matrix which satisfy the constant unit sum of every row transition probabilities. This transition matrix in (13) also satisfy the model assumptions stated earlier in this section. We now apply the model in the following section.

3.0 Practical Application of the Model

Using five-year (2000-2004) data obtained from XYZ University in Nigeria, the transition matrix in (14), which is equivalent to that in equation (13) was obtained. The actual name of the University in Nigeria is being withheld because of the confidential consideration of the University. In collecting the data from XYZ University, the various ranks of academic staff were classified and denoted as follow:

Ju - Junior Lecturers made up of Asst. Lecturers, Lecturers II and Lecturers I.
Sn - Senior Lecturers
As - Associate Professors
Pr - Professors
LBRD - Those who could leave by resignation/dismissal
LBRC - Those who could leave by retirement/contract appointment.

The summary of the five-year data is shown in Table 1.

Table 1: Showing the total number of XYZ university academic staff in each cadre who either remained in that cadre or promoted to the next cadre or entered an absorbing state from the year 2000 to 2004.

Rank	Ju	Sn	As	Pr	LBRD	LBRC	No. in Each Nonabsorbing rank
Ju	353	167	0	0	99	0	619
Sn	0	103	31	0	47	0	181
As	0	0	57	3	10	0	70
Pr	0	0	0	38	0	2	40

The following transition matrix was derived from Table 1.

		Transient States				Absorbing States	
		Ju	Sn	As	Pr	LBRD	LBRC
Transient States	Ju	0·57	0·27	0·0	0·0	0·16	0·0
	Sn	0·0	0·57	0·17	0·0	0·26	0·0
	As	0·0	0·0	0·81	0·05	0·14	0·0
	Pr	0·0	0·0	0·0	0·96	0·0	0·04
Absorbing States	LBRD	0·0	0·0	0·0	0·0	1·0	0·0
	LBRC	0·0	0·0	0·0	0·0	0·0	1·0

(14)

(14) is the partitioned transition matrix of Academic staff Promotion in XYZ University.
The only difference between the stationary matrix in (13) and that of (14) is the inclusion of absorbing states in (14). A state e is said to be absorbing if $p_{ee} = 1$, that is a state you enter and you cannot return to any other state. Consequently, the Markov chain of transition matrix in (13) is said to be irreducible while that of (14) is not irreducible.

Let the transition matrix in (14) be denoted by P and the kth step transition matrix be denoted by P^k. Then

$$p^2 = \begin{pmatrix} 0\cdot3249 & 0\cdot3078 & 0\cdot0459 & 0 & 0\cdot3214 & 0 \\ 0 & 0\cdot3249 & 0\cdot2346 & 0\cdot0085 & 0\cdot4320 & 0 \\ 0 & 0 & 0\cdot6561 & 0\cdot0885 & 0\cdot2534 & 0\cdot0020 \\ 0 & 0 & 0 & 0\cdot9216 & 0 & 0\cdot0784 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$p^5 = \begin{pmatrix} 0\cdot0602 & 0\cdot1425 & 0\cdot1290 & 0\cdot0123 & 0\cdot6557 & 0\cdot0004 \\ 0 & 0\cdot0602 & 0\cdot2044 & 0\cdot0416 & 0\cdot6915 & 0\cdot0024 \\ 0 & 0 & 0\cdot3487 & 0\cdot1556 & 0\cdot4799 & 0\cdot0158 \\ 0 & 0 & 0 & 0\cdot8154 & 0 & 0\cdot1846 \\ 0 & 0 & 0 & 0 & 1\cdot0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1\cdot0000 \end{pmatrix}$$

and

$$\underset{k\to\infty}{Lt}\, P^k = \begin{pmatrix} 0 & 0 & 0 & 0 & 0\cdot935 & 0\cdot065 \\ 0 & 0 & 0 & 0 & 0\cdot896 & 0\cdot104 \\ 0 & 0 & 0 & 0 & 0\cdot737 & 0\cdot263 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (15)$$

Equation (15) are the transition probabilities after a large number of steps. The results in (15) are discussed in section 4.0.

4.0 DISCUSSION OF RESULTS

A careful observation of the first quadrant part of the partition matrix in (15) shows that in XYZ University the Junior lecturers have very high percentage (93.5%) of voluntary resignation or dismissal. Some of the possible reasons that were corroborated by our findings in XYZ University include: (1) some of the Junior lecturers may not be settled for a permanent career in terms of job satisfaction. (2) some junior lecturers may exhibit youthful exuberance which can consequently lead to poor performance resulting in dismissal. These happenings reduce as we climb the ladder when these attributes commonly found with the lower cadres are rarely found (in fact 0% in this illustration) with professors at the top of the ladder. However, with up to 93.5% of junior lecturers that may opt for resignation/dismissal absorbing state, it should be of great concern to review some frustrating entering and promotion criteria that are facing staff at that cadre.

In the case of retirement/contract appointment absorbing state in XYZ University, 6.5% of the junior lecturers are likely to opt for retirement/contract appointment. The few junior lecturers that may do so might have transferred their many years services from other nonacademic professions (without having the needed research publications) to XYZ University and found appointable as Junior Lecturers. Conversely, 100% of the professorial cadre can retire or take up contract appointment after retirement. Any academic staff who has risen to the rank of a professor should have put in the minimum years required to merit retirement. Hence the results in (14) explain what practically goes on in XYZ University.

5.0 CONCLUSION

The transition matrix of the staff promotion Markov chain model has been derived using probability generating function (PGF) and applied to academic staff promotion in a University. This has led to results that have assisted in getting better insight into the entering point and promotion criteria of a University with the need to initiate a review process.

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REFERENCES

Carey, K. J., and Sherr, L. A., 1974, Market and Price Factors in Transaction-to- Transaction Price Change Behaviour of Common Stocks, Applied Economics, Vol. 6, pp. 21-26.

Davies, R., Johnson, D., and Farrow, S., 1975. Planning Patient Care with a Markov Model, Operational Research Quarterly, Vol. 26. pp. 44-50.

Dinkel, J .J., Kochenberger G. A., and Plane, D. R., 1978, Management Science Text and Applications, Homewood: Richard D. Irwin, Inc.

Gani, J., 1991. Generating Function Methods in a Random Allocation Problem of Epidemics, Bull. Inst. Comb. Appl. 3, 43-50.

Gani, J., 1993. Random Allocation Methods in an Epidemic Model in Stochastic Processes, New York: Springer Inc.

Gani, J., 2002a. Needle Sharing Infections Among Heterogeneous IVDUs, Monatsh Math. 135, 25-36.

Gani, J., 2002b. Nonhomogeneous Susceptibles and Infectives Among Needle Sharing IVDUs, Singapore: World Scientific, 91-99.

Gani, J., 2003. Random Allocation and Urn Models. Available online at: Http//www.maths.anu.edu.au.

Gani, J. and Yakowitz, S., 1993. Modelling the Spread of HIV Among Intraveneous Drug Users. IMA J. Maths. Appl. Med Biology 10, 51-65.

Gregory, G., and Litton, C. D., 1975. A Markov Analysis of a Single Conveyor System, Management Science, Vol. 22, pp. 8-18.

Grinold, R. C., 1976. Manpower Planning with Uncertain Requirements, Operations Research, Vol. 24, pp. 32-36.

Rutherford, R. D., 1954. On a Contageous Distribution, Ann. Maths. Statist. 25, 703-713.

Schachtman, R. H. and Hogue, C. J., 1976, Markov Chain Model for Events Following Induced Abortion, Operations Research, Vol. 24, pp. 7-12.

Simmons, D. M., 1971. Common-Stock Transaction Sequences and the Random-Walk Model, Operations Research, Vol. 19, pp. 40-47.

Valliant, R., and Milkovich, G. T., 1977. Comparison of Semi-Markov and Markov Models in a Personnel Forecasting Application, Decision Sciences, Vol. 8, pp. 18-23.