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AUTOCORRELATION FUNCTIONS AND THE JUSTIFICATION OF THE ARMA TRANSFORM OF THE GARCH MODEL EQUATION

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ABSTRACT

We derived the theoretical moments and autocorrelation functions of GARCH models and those of their ARMA transform. The autocorrelation structures are found to be the same for the two models. On the basis of this, we conclude that the ARMA transform is appropriate for GARCH models.

KEYWORDS: ARMA, GARCH, ARCH, ARMA transform

1.0 INTRODUCTION

The assumption of constant variance in the traditional time series models of Autoregressive Moving Average Models(ARMA) is a major impediment to their applications in financial time series data where heteroscedasticity is obvious and cannot be neglected.

To solve the stated problem, Engle (1982) proposed Autoregressive Conditional Heteroscedasticity (ARCH) model. However, Engle in his first application of ARCH noted that a high order of ARCH is needed to satisfactorily model time varying variances. It is noted that many parameters in ARCH will create convergence problems for maximization routines see for example Bollerslev (1986). To avoid these problems, Bollerslev (1986) extended Engle's model to Generalized Autoregressive Conditional Heteroscedasticity models (GARCH). This models time-varying variances as a linear function of past square residuals and of its past value. It has proved useful in interpreting volatility clustering effects and has wide acceptance in measuring the volatility of financial markets. The ARCH and GARCH models are known as symmetric models see Nelson (1991) for example.

Other extensions are the exponential GARCH (EGARCH) model of Nelson (1991), the model of Glosten, Jaganathan and Runkle (GJR-GARCH) of 1993 as well as the threshold model (TGARCH) of Zakoian (1994). These model and interpret leverage effect, where volatility is negatively correlated with returns. Equally important is The Fractionally Integrated GARCH model (FIGARCH) of Baillie, Bollerslev and Mikeson (1996) which is introduced to model long memory via the fractional operator $(1\text{-}L)^d$.

It is customary in literature to transform the GARCH model through $a_t = \varepsilon_t^2 - h_t$ to an ARMA model see Karanasos and kim (2001) for example. The aim of this paper is to attempt the justification of this practice.

The approach is by comparing the autocorrelation functions of the GARCH model with that of the ARMA transform. Eni and Etuk(2006) have used the same approach to justify the Autoregression transform of the ARCH model equation.

2.0 THE GARCH (p,q) MODEL

To make for parsimony in the modeling of conditional heteroscedasticitiy, Bollerslev (1986) proposed the generalized ARCH model denoted GARCH (p,q) model.
In a GARCH model, the conditional variance is presented as a linear function of past squared returns and of its past value. That is

$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} B_j h_{t-j} \quad \text{.......} \quad 2.1$$

with parametric constraints

$$\alpha_0 > 0; \alpha_i \geq 0 \qquad i = 1,\cdots,q; B_j \geq 0 \quad j = 1,\cdots,p.$$

If p=0, then (2.1) is an ARCH(q) process .and if p= q=0, then h_t is constant.

(2.1) can be written in the form

$$h_t = \alpha_0 + \alpha(L)\varepsilon_t^2 + B(L)h_t \quad \text{.......} \quad 2.2$$

where

$\alpha(L) = \alpha_1 L + \alpha_2 L^2 \cdots L^q$ and $B(L) = B_1 L + B_2 L^2 \cdots Bp L^p$

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Further more, re-writing (2.2) as

$$(1-B(L))h_t = \alpha_0 + \alpha(L)\varepsilon_t^2$$

$$h_t = \frac{\alpha_0}{1-B(L)} + \frac{\alpha(L)}{1-B(L)}\varepsilon_t^2$$

$$= \frac{\alpha_0}{1-B(L)} + \sum_{i=1}^{\infty} \Lambda_i \varepsilon_{t-i}^2 \qquad \cdots 2.3$$

where Λ_i is the coefficient of L^i in the Taylor series expansion of

$$\alpha(L)(1-B(L))^{-1},$$

which is an infinite ARCH model.

The GARCH (p,q) model is related to the ARMA(p,q) model through the substitution of $h_t = \varepsilon_t^2 - a_t$ to get

$\varepsilon_t^2 = \alpha_0 + (\alpha_1 + B_1)\varepsilon_{t-1}^2 - B_1 a_{t-1} + a_t$

which is an ARMA(Max(p,q),q) model. This relation suggests that the theory underlying time series ARMA models can be applied to GARCH models.

3.0 MOMENTS AND AUTOCORRELATION OF GARCH MODEL

Proposition I: The Mth moment of GARCH (1,1) model is

$$E(\varepsilon_t^{2m}) = E(Z^{2m})E\left[\frac{\sum_{i=1}^{m} \frac{\Gamma(m+1)}{\Gamma(m+1-i)\Gamma(i+1)} \alpha_0^i \left(\alpha_1 Z^2 + B_1 \frac{\varepsilon_{t-1}^2}{Z^2}\right)^{m-i}}{1-E(Z^2\alpha_1 + B_1)^m}\right] \qquad \ldots\ldots 3.1$$

where $\Gamma(.)$ is a gamma function

Proof

$$h_t = \alpha_0 + \alpha_1\varepsilon_{t-1}^2 + B_1 h_{t-1}$$

$$= \alpha_0 + (\alpha_1 Z^2 + B_1)h_{t-1}$$

$$h_t^m = \sum_{i=0}^{m} \frac{\Gamma(m+1)}{\Gamma(m+1-i)\Gamma(i+1)} \alpha_0^i \{(\alpha_1 Z^2 + B_1)h_{t-1}\}^2$$

$$E(\varepsilon_t^{2m}) = E(Z^{2m})E\left[\frac{\left[\sum_{i=1}^{m} \frac{\Gamma(m+1)}{\Gamma(m+1-i)\Gamma(i+1)} \alpha_0^i \left[(\alpha_1 Z_1^2 + B_1)\frac{\varepsilon_t^2}{Z^2}\right]\right]^{m-i}}{1-E(\alpha_1 Z^2 + B_1)^m}\right]$$

Remark

We note that for

$$E(\varepsilon_t^{2m}) < \infty \quad ,\alpha_1 + B_1 < 1$$

This becomes the necessary and sufficient condition for stationarity. We note also that for

$$E(\varepsilon_t^{2m}) < \infty \ , \quad E(\log(\alpha_1 Z^2 + B_1)) < 0$$

This condition is necessary and sufficient for strict stationarity and Ergodicity of h_t .It is also in agreement with Nelson (1991).Since it allows the case of $\alpha_1 + B_1$. The condition $E(\log(\alpha_1 Z^2 + B_1)) < 0$ is weaker than that of $\alpha_1 + B_1 < 1$

We also note that the presence of $E\left(\varepsilon^{2(m-1)}\right)$ in the numerator (3.1) suggests that the $(m-1)_{th}$ moment must exist for the m_{th} moment to be well defined.

Corollary

i $$E\left(\varepsilon_t^2\right)=\frac{\alpha_0}{1-E\left(\alpha_1+B_1\right)} \qquad \cdots 3.2$$

ii $$E\left(\varepsilon_t^4\right)=\frac{E\left(Z^4\right)\alpha_0^2\left[E\left(\alpha_1 Z^2+B_1\right)+1\right]}{\left[1-E\left(\alpha_1 Z^2+B_1\right)\right]\left[1-E\left(\alpha_1+B_1\right)^2\right]} \qquad \cdots 3.3$$

iii $$V(\varepsilon_t^2)=\frac{\alpha_0^2\left[E(Z_t^4)\left[1-\left(E\left(\alpha_1 Z^2+B_1\right)\right)^2\right]-\left[1-E\left(\alpha_1 Z^2+B_1\right)^2\right]\right]}{\left[1-E\left(\alpha_1 Z_t^2+B_1\right)\right]^2\left[1-E\left(\alpha_1 Z^2+B_1\right)^2\right]} \qquad \ldots 3.4$$

Proof

Case i is elementary

Case ii

Substituting m=2 into (3.1), we have

$$E\left(\varepsilon^4\right)=E\left(Z^4\right)\frac{\left[2\alpha_0\left(\alpha_1 Z^2+B_1\right)\frac{\varepsilon^2}{Z_t^2}+\alpha_0^2\right]}{1-E\left(\alpha_1 Z^2+B_1\right)}$$

$$=\frac{E\left(Z^4\right)\left[2\alpha_0^2 E\left(\alpha_1 Z^2+B_1\right)+\alpha_0^2\left[1-E\left(Z^2\alpha_1+B_1\right)\right]\right]}{\left[1-E\left(Z^2\alpha_1+B_1\right)\right]\left[1-E\left(\alpha_1 Z^2+B_1\right)^2\right]}$$

$$=\frac{E\left(Z_t^4\right)\alpha_0^2\left[E\left(\alpha_1 Z^2+B_1\right)+1\right]}{\left[1-E\left(\alpha_1 Z^2+B_1\right)\right]\left[1-E\left(\alpha_1+B_1\right)^2\right]}$$

Case iii

Proof

$$V\left(\varepsilon_t^2\right)=E\left(\varepsilon_t^4\right)-\left[E\left(\varepsilon_t^2\right)\right]^2$$

$$=\frac{E\left(Z_t^4\right)\alpha_0^2\left[E\left(\alpha_1 Z^2+B_1\right)+1\right]}{\left[1-E\left(\alpha_1 Z_t^2+B_1\right)\right]\left[1-E\left(\alpha_1 Z^2+B_1\right)^2\right]}-\frac{\alpha_0^2}{\left[1-E\left(\alpha_1 Z^2+B_1\right)\right]^2}$$

$$=\frac{\alpha_0^2\left[E\left(Z_t^4\right)\left[1-\left(E\left(\alpha_1 Z^2+B_1\right)\right)^2\right]-\left[1-E\left(\alpha_1 Z^2+B_1\right)^2\right]\right]}{\left[1-E\left(\alpha_1 Z_t^2+B\right)\right]^2\left[1-E\left(\alpha_1 Z^2+B\right)^2\right]}$$

Remarks

By substitution of $E\left(Z_t^4\right)$, it is easy to see that under condition of normality

:

$$V\left(\varepsilon_t^2\right)=\frac{2\alpha_0^2\left[1-2\alpha_1 B_1-B_1^2\right]}{\left[1-\left(\alpha_1+B_1\right)\right]^2\left[1-3\alpha_1^2-2\alpha_1 B_1-B_1^2\right]} \qquad \cdots 3.5$$

Hence the conditions for a positive variance are $\alpha+B<1$ and $3\alpha_1^2-2\alpha_1 B-B_1^2<1$.

Proposition 11

The autocovariance between ε_t^2 *and* ε_{t-n}^2 of a GARCH(1, 1) model

$$\operatorname{cov}(\varepsilon_t^2\varepsilon_{t-n}^2) = V_n =$$

$$\alpha_0^2\left[\frac{[1-E(A)][1-E(A)^2\sum_{i=0}^{n-1}B_1{}' + B''[1-E(A)]^2-(1-E(A)^2)}{\{1-E(A)\}^2[1-E(A)^2]} + \frac{[1-E(A)]^2[1-E(A)^2\sum_{i=0}^{n-1}\alpha_1B_1{}'E((\varepsilon_{t-i-1}^2\varepsilon_{t-n}^2))}{\{1-E(A)\}^2[1-E(A)^2]}\right]$$

where A= $(Z_t^2\alpha_1 + B_1)$ ...3 .6

Proof

There are two parts of the proof. In the first part we find expression for $E(\varepsilon_t^2\varepsilon_{t-n}^2)$ while in the second part, we find var $(\varepsilon_t^2\varepsilon_{t-n}^2)$

Part 1

$$h_t = \alpha_0 + \alpha_1\varepsilon_{t-1}^2 + B_1h_{t-1}$$

$$h_th_{t-1} = (\alpha_0 + \alpha_1\varepsilon_{t-1}^2 + B_1h_{t-1})h_{t-1}$$

replacing $h_{t-1} = \alpha_0 + \alpha_1\varepsilon_{t-2}^2 + B_1h_{t-2}$

We have

$$h_th_{t-2} = (\alpha_0 + B_1\alpha_0 + \alpha_1\varepsilon_{t-1}^2 + \alpha B_1\varepsilon_{t-2}^2 + B_1^2h_{t-2})h_{t-2}$$

After repeated recursions, we have

$$h_th_{t-n} = \sum_{i=0}^{n-1}\alpha_0B_1^ih_{t-n} + \sum_{i=0}^{n-1}\alpha_0B_1^i\varepsilon_{t-i-1}^2h_{t-n} + B_1^nh_{t-n}^2$$

$$\varepsilon_t^2\varepsilon_{t-n}^2 = Z_t^2\sum\alpha_0B_1^i\varepsilon_{t-n}^2 + Z_t^2\sum_{i=0}^{n-1}\alpha_0B_1^i\varepsilon_{t-i-1}^2\varepsilon_{t-n}^2 + Z_t^2Z_t^2B_1^n\frac{\varepsilon_{t-n}^4}{Z_{t-n}^4}$$

Taking expectations, we have

$$E(\varepsilon_t^2\varepsilon_{t-n}^2) = \frac{\alpha_0^2\sum_{i=0}^{n-1}B_1{}'}{1-E(A)} + \sum_{i=0}^{n-1}\alpha_1B_1{}'E(\varepsilon_t^2\varepsilon_{t-n}^2) + \frac{B_1{}''\alpha_0^2(1-E(A))}{(1-E(A))(1-E(A)^2)}$$

$$= \frac{\alpha_0^2[\,[1-E(A)^2\sum_{i=0}^{n-1}B'\,] + B_1{}''[1+E(A)]^2 - [\,1-E(A)^2]\,]}{[1-E(A)\,][\,1-E(A)^2]} + \frac{[1-E(A)][1-E(A)^2\sum_{i=0}^{n-1}\alpha_1B_1{}'E(\varepsilon_t^2\varepsilon_{t-n}^2)}{[1-E(A)\,][\,1-E(A)^2]}$$

Part II
By definition

$$V_n = Cov(\varepsilon_t^2 \varepsilon_{t-n}^2) =$$

$$\frac{\alpha_0^2\left[\ [1-E(A)^2 \sum_{i=0}^{n-1} B' \ ] + B_1''[1+E(A)]\ \right]}{[1-E(A)][1-E(A)^2]} + \frac{[1-E(A)][1-E(A)^2 \sum_{i=0}^{n-1} \alpha_1 B_1' E(\varepsilon_t^2 \varepsilon_{t-n}^2)}{[1-E(A)][1-E(A)^2]} - \frac{\alpha_0^2}{[1-E(A)^2]}$$

$$= \frac{\alpha_0^2\left[\{1-E(A)\}[1-E(A)^2 \sum_{i=0}^{n-1} B' + B''\alpha_0^2[1-E(A)]^2 - \alpha_0^2(1-E(A)^2)\right] + \left[\ [1-E(A)^2][1-E(A)]^2 \sum_{i=0}^{n-1} \alpha_1 B_1' E(\varepsilon_{t-i-1}^2 \varepsilon_{t-n}^2)\ \right]}{[1-E(A)]^2[1-E(A)^2]}$$

Corollary
Under normality assumptions of Z_t

(i) (i) $$Cov(\varepsilon_t^2 \varepsilon_{t-1}^2) = V_1 = \frac{2(\alpha_1 - \alpha_1 B_1^2 - \alpha_1 B_1^2)\alpha_0^2}{(1-\alpha_1 - B_1^2)^2(1-3\alpha_1^2 - 2\alpha_1 B_1^2 - B_1^2}$$ ...3.7

(ii) $$Cov(\varepsilon_t^2 \varepsilon_{t-1}^2) = V_2 = \frac{2(\alpha_1 + B_1)(\alpha_1 - \alpha_1 B_1^2 - \alpha_1 B_1^2)\alpha_0^2}{(1-\alpha_1 - B_1^2)^2(1-3\alpha_1^2 - 2\alpha_1 B_1^2 - B_1^2)}$$ ...3.8

iii And in general

$$\rho(\varepsilon_t^2 \varepsilon_{t-n}^2) = \frac{[1-E(A)][1-E(A)^2]\sum_{i=0}^{N-1} B_1' + [1-E(A)^2[1-E(A)]^2 \frac{1}{\alpha_0}\sum_{i=0}^{N-1}\alpha_i B_i' E(\varepsilon_{t-n-1}^2 \varepsilon_{n-n}^2) - B_1''[1-E(A)]^2 - (1-E(A)^2)}{E(Z^4)[1-E(A)]}$$ ... 3.9

Proof
Case 1. Using earlier results

$$V_1 = \frac{\left[\begin{array}{l}(1-\alpha_1 - B_1)(1-3\alpha_1^2 - 2\alpha_1\beta_1 - B_1^2) + \\ B[1-\alpha_1^2 - 2\alpha_1 B_1 - B_1^2] - (1-3\alpha_1^2 - 2\alpha_1 B_1 - B_1^2)\end{array}\right]\alpha_0^2 + \alpha_1(1-\alpha_1 - B_1)^2(1-3\alpha_1^2 - 2\alpha_1 B_1 - B_1^2)E(\varepsilon_t^4)}{(1-\alpha_1 - B_1)^2(1-3\alpha_1^2 - 2\alpha_1 B_1 - B_1^2)}$$

$$V_1 = \alpha_0^2\left\{\frac{(1-\alpha_1 - B_1)(1-3\alpha_1^2 - 2\alpha_1\beta_1 - B_1^2) + B_1[1-\alpha_1^2 - 2\alpha_1\beta_1 - B_1^2] - (1-3\alpha_1^2 - 2\alpha_1 B_1 - B_1^2) + 3\alpha_1(1-3\alpha_1^2 - 2\alpha_1\beta_1 - B_1^2)}{(1-\alpha_1 - B_1)^2(1-3\alpha_1^2 - 2\alpha_1 B_1 - B_1^2)}\right\}$$

$$V_1 = \frac{\left\{\begin{matrix} -\alpha_1 + 3\alpha_1^3 + 2\alpha_1^2 B_1 + \alpha_1 B_1^2 - B_1 + 3\alpha_1^2 B_1 + 2\alpha_1 B_1^2 + B_1^3 + 3\alpha_1 - 3\alpha_1^3 \\ -6\alpha_1^2 B_1 - 3\alpha_1 B_1^2 + B_1 - B_1 \alpha_1^2 - 2\alpha_1 B_1^2 + B_1^3 \end{matrix}\right\}\alpha_0^2}{(1-\alpha_1 - B_1)^2(1 - 3\alpha_1^2 - 2\alpha_1 B_1 - B_1^2)}$$

This reduces to

$$V_1 = \frac{(2\alpha_1 - 2\alpha_1^2 B - 2\alpha_1 B_1^2)\alpha_0^2}{(1-\alpha_1 - B_1)^2(2 - 3\alpha_1^2 - 2\alpha_1 B_1 - B_1^2)}$$

$$= \frac{2(\alpha_1 - \alpha_1^2 B - 2\alpha_1 B^2)\alpha_0^2}{(1-\alpha_1 - B_1)^2(1 - 3\alpha_1^2 - 2\alpha_1 B_1 - B_1^2)}$$

Case 11

$V_2 =$

$$\frac{\alpha_0^2\left[\begin{matrix} (1-\alpha_1 - B_1)(1-3\alpha_1^2 - 2\alpha_1 B_1 - B_1^2)(1 + B_1) + 2\alpha_1(\alpha_1 - \alpha_1 B_1^2 - \alpha^2 B_1) + \\ \alpha_1(1 - 3\alpha_1^2 - 2\alpha_1 B_1 - B_1^2) + 3\alpha B(1 - \alpha_1^2 - 2\alpha_1 B_1 - B_1^2 \\ + B_1^2[1 - \alpha^2 - 2\alpha_1 B_1 - B^2] - 1 + 3\alpha_1^2 + 2\alpha_1 B_1 + B_1^2 \end{matrix}\right]}{(1-\alpha_1 - B_1)^2(1 - 3\alpha_1^2 - 2\alpha_1 B_1 - B^2)}$$

$$\frac{\alpha_0^2\left[\begin{matrix} 1 - 3\alpha_1^2 - 2\alpha_1 B_1 - B_1^2 - \alpha + 3\alpha_1^3 + 2\alpha_1^2 B_1 + \alpha_1 B_1^2 - B_1 \\ + 3\alpha_1^2 B_1 + 2\alpha_1 B_1^2 + B_1^3 + B_1 - 3\alpha_1^2 B_1 - 2\alpha_1 B_1^2 + B_1^3 - \alpha_1 B_1 \\ + 3\alpha_1^2 B + 2\alpha_1^2 B_1^2 + \alpha_1 B_1^3 - B_1^2 + 3\alpha_1^2 B_1^2 + 2\alpha_1 B_1^3 + B_1^4 \\ + 2\alpha_1^2 - 2\alpha_1^2 B_1^2 - 2\alpha_1^3 B_1 + \alpha_1 - 3\alpha_1^3 - 2\alpha_1^2 B - \alpha_1 B_1^2 \\ + 3\alpha_1 B_1 - 3\alpha_1^3 B_1 - 6\alpha_1^3 B_1^2 - 3\alpha_1 B_1^3 + B_1^2 - \alpha_1^2 B_1^2 \\ - 2\alpha_1 B_1^3 - B_1^4 - 1 - 3\alpha_1^2 - 2\alpha_1 B_1 + B^2 \end{matrix}\right]}{(1-\alpha_1 - B_1)^2(1 - 3\alpha_1^2 - B_1 - B_1^2)}$$

This reduces to

$$V_2 = \frac{2(\alpha_1 + B_1)(\alpha_1 - \alpha_1 B_1^2 - \alpha_1^2 B_1)}{(1-\alpha_1 - B_1)^2((1 - 3\alpha_1^2 - B_1^2 - 2\alpha B^2)}$$

We can conclude that in general

$$V_n = \frac{2(\alpha_1 + B_1)^{n-1}(\alpha_1 + \alpha_1 B_1^2 - \alpha_1^2 B_1)}{(1-\alpha_1 - B_1)^2(1 - 3\alpha_1^2 - 2\alpha_1 B_1 - B_1^2)}$$

or 3.10

$$V_n = (\alpha_1 + B_1)^{n-1} V_1$$

Case iii

Proof

Using $\rho(\varepsilon_t^2 \varepsilon_{t-n}^2) = \frac{V_n}{V_0}$ in (3, 3) and (3.10) the result becomes obvious

Remark,

it is easy to see that under normality assumption

$$\rho_1 = \frac{(\alpha_1 - \alpha_1^2 B_1 - \alpha_1 B^2)}{1 - 2\alpha_1 B_1 - B_1^2} \quad \ldots 3.11$$

$$\rho_2 = \frac{(\alpha_1 + B_1)(\alpha_1 - \alpha_1 B_1^2 - \alpha_1^2 B_1)}{1 - 2\alpha_1 B_1 - B_1^2} \quad \ldots 3.12$$

And in general

$$\rho_n = \frac{(\alpha_1 + B_1)^{n-1}(\alpha_1 + \alpha_1 B_1^2 - \alpha_1^2 B_1)}{1 - 2\alpha_1 B_1 - B_1^2} \quad \ldots 3.13$$

$$\rho_n = (\alpha_1 + B_1)^{n-1}\rho_1 \qquad n = 2,3\ldots$$

4.0 RELATIONSHIP WITH ARMA MODELS

As already discussed in section 2.0,GARCH (p,q) models admits transformations to ARMA(p,q) models through the substitution

$$h_t = \varepsilon_t^2 - a_t$$

Hence GARCH(1,1) model becomes

$$\varepsilon_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + B_1(\varepsilon_{t-1}^2 - a_{t-1}) + a_t$$

$$\varepsilon_t^2 = \alpha_0 + (\alpha_1 + B_1)\varepsilon_{t-1}^2 - B_1 a_{t-1} + a_t \qquad (i) \qquad \cdots 4.1$$

This is an ARMA (1,1) model

Multiplying through by ε_t^2, ε_{t-1}^2, we have

$$E_0 = \alpha_0 E(\varepsilon_t^2) + (\alpha_1 + B_1)E_0 - B_1 E(\varepsilon_t^2 a_{t-1}) + \sigma_a^2 \qquad (ii)$$

$$E_1 = \alpha_0^2 E(\varepsilon_t^2) + (\alpha_1 + B_1)E_1 - B_1\sigma_a^2 \qquad (iii)$$

To find $E(\varepsilon_t^2 a_{t-1})$, we multiply by (i) by a_{t-1} to get

$$E(\varepsilon_t^2 a_{t-1}) = (\alpha_1 + B_1)\sigma_a^2 - B_1\sigma_a^2 = \alpha_1\sigma_a^2$$

Hence (ii) becomes

$$E_0 = \alpha_1 E(\varepsilon_t^2) + (\alpha_1 B_1)E_0 + (1 - \alpha_1 B_1)\sigma_a^2 \qquad (iv)$$

Solving (iii) and (iv) simultaneously for E_0, we have

$$E_0 = \frac{\alpha_0^2}{(1 - \alpha_1 B_1)^2} + \frac{(1 - 2\alpha_1 B_1 - B_1^2)\sigma_a^2}{1 - (\alpha_1 + B_1)^2}$$

Hence

$$Var(\varepsilon_t^2) = V_0 = \frac{(1 - 2\alpha_1 B_1 - B_1^2)\sigma_a^2}{1 - (\alpha_1 + B_1)^2}$$

Also

$$E_1 = \frac{\alpha_0^2}{(1 - \alpha_1 - B_1)^2} + \frac{(\alpha_1 - \alpha_1^2 B_1 - \alpha_1 B_1^2)\sigma_a^2}{1 - (\alpha_1 + B_1)^2}$$

And

$$Cov(\varepsilon_t^2 \varepsilon_{t-1}^2) = V_1 = \frac{(\alpha_1 - \alpha_1^2 B_1 - \alpha_1 B_1^2)\sigma_a^2}{1-(\alpha_1 + B_1)^2}$$

$$E_2 = \frac{\alpha_0^2}{(1-\alpha_1 + B_1)} + (\alpha_1 + B_1)E_1$$

Also,

$$E_3 = \frac{\alpha_0^2}{(1-\alpha_1 - B_1)} + (\alpha_1 + B_1)^2 E_1$$

And in general

$$E_n = \frac{\alpha_0^2}{(1-\alpha_1 - B_1)} + (\alpha_1 + B_1)^{n-1} E_1$$

Or

$$V_2 = (\alpha_1 + B_1)E_1$$

$$V_3 = (\alpha_1 + B_1)^2 E_1$$

$$\vdots$$

$$V_n = (\alpha_1 + B_1)^{n-1} E_1$$

Hence the autocorrelation functions become

$$\rho_1 = \frac{\alpha_1 - \alpha_1^2 B_1 - \alpha_1 B_1^2}{1 - 2\alpha_1 B_1 - B_1^2} \qquad 4.2$$

$$\rho_2 = \frac{(\alpha_1 + B_1)^2(\alpha_1 - \alpha_1^2 B_1 - \alpha_1 B_1^2)}{1 - 2\alpha_1 B_1 - B_1^2} \qquad 4.3$$

$$\vdots$$

$$\rho_n = (\alpha_1 + B_1)^{n-1}\rho_1 \qquad 4.4$$

5.0 CONCLUSION

The results in 4.2, 4.3, and 4.4 are in agreement with 3.11, 3.12 and 3.13. we conclude that 4.1 is a proper transformation of 2.1 for p=q=1 These results suggests that characteristics behavior of time series ARMA (p,q) models can be applied to GARCH(p,q) models.

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