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CONVOLUTION PROPERTIES ASSOCIATED WITH CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

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ABSTRACT

Murugusundaramoorthy and Magesh (2004) introduced the subclasses $TS(\alpha,\beta)$ and $TS_b(\alpha,\beta)$ of uniform convex functions and starlike functions with negative coefficients where they obtained some results. Our aim here is to investigate the convolution properties associated with the subclasses $TS^*(\alpha,\beta)$ and $TS_b^*(\alpha,\beta)$ respectively by applying certain techniques based especially upon the Cauchy-Schwarz and Holder inequalities. Some consequences are also discussed.

KEY WORDS: Analytic, Convolution, Convex, Starlike, Univalent.

1. INTRODUCTION, DEFINITIONS AND PRELIMINARY RESULTS.

Denoted by S the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad 1.1$$

that are analytic and univalent in the unit disk $E = \{z : |z| < 1\}$ .Also, ST and CV are the subclasses of S_n that are respectively starlike and convex.

A function is uniformly convex(uniformly starlike) in E if $f(z)$ is in CV(ST) and has the property that for every circular arc γ contained in E, with centre ε also in E, the arc $f(\gamma)$ is convex(starlike) with respect to $f(\varepsilon)$. The class of uniformly convex functions is denoted by UCV and the class of uniformly starlike functions by UST.
It is well known that

$$f \in UCV \Leftrightarrow \left|\frac{zf''(z)}{f'(z)}\right| \le \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\}$$

see Ma and Minda (1992). Ronning (1993) introduced a new subclass of starlike functions related to UCV defined as

$$f \in S_p \Leftrightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| \le \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}.$$

We note here that $f(z) \in UCV \Leftrightarrow zf'(z) \in S_p$.

Later, Ronning (1993) generalized the class S_p by introducing a parameter α, $-1 \le \alpha < 1$,

$$f \in S_p(\alpha) \Leftrightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| \le \operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\}$$

Also Murugusundaramoorthy and Magesh (2004) introduced subclasses $TS(\alpha,\beta)$ and $TS_b(\alpha,\beta)$.

Here we let $TS(\alpha,\beta) = S(\alpha,\beta) \cap T$ where T, the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \ge 0, \ \forall n \ge 2. \qquad 1.2$$

This class of functions was introduced and studied by Silverman(1995), Silverman and Silvia (1997). And also let $TS_b(\alpha,\beta)$ denote the class of function, $f(z)$ in $TS(\alpha,\beta)$ and be of the form

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$$f(z)=z-\frac{b(1-\alpha)(c)}{(2+\beta-\alpha)(a)}z^2-\sum_{n=3}^{\infty}a_nz^n \qquad (a_n\geq 0 \quad 0\leq b\leq 1). \tag{1.3}$$

The main aim of this paper is to investigate the convolution properties associated with the subclasses $TS^*(\alpha,\beta)$ and $TS_b^*(\alpha,\beta)$.

A necessary and sufficient condition for a function $f(z)$ of the form (1.2) to be in the class $TS^*(\alpha,\beta)$, $-1\leq\alpha<1, \beta\geq 1$ is that

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}}a_n\leq 1-\alpha$$

Also, $TS_b^*(\alpha,\beta)$ denote the class of functions $f(z)$ in $TS^*(\alpha,\beta)$ and of the form

$$f(z)=z-\frac{b(1-\alpha)(c)}{(2+\beta-\alpha)(a)}z^2-\sum_{n=3}^{\infty}a_nz^n \quad (a_n\geq 0), 0\leq b\leq 1$$

if and only if

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}}a_n\leq(1-b)(1-\alpha) \qquad -1\leq\alpha<1, \beta\geq 0$$

To do this we need the following preliminary results which we shall state without proof.

Theorem A; Murugusundaramoorthy and Magesh (2004). A necessary and sufficient condition for $f(z)$ of the form (1.2) to be in the class $TS(\alpha,\beta)$, $-1\leq\alpha<1$, $\beta\geq 0$ is that

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}}a_n\leq 1-\alpha \tag{1.4}$$

Theorem B; Murugusundaramoorthy and Magesh (2004). Let function $f(z)$ be defined by (1.3) then $f(z)\in TS_b(\alpha,\beta)$ if and only if

$$\sum_{n=3}^{\infty}[n(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}}a_n\leq(1-b)(1-\alpha) \tag{1.5}$$

Finally, for functions $f_j(z)\in S$ $(j=1,....,m)$ given by

$$f_j(z)=z-\sum_{n=2}^{\infty}a_{n,j}z^n \qquad (j=1,....,m) \tag{1.6}$$

the Hadamard product(or convolution) is defined by

$$(f_1*...f_m)(z)=z-\sum_{n=2}^{\infty}\left(\prod_{j=1}^{m}a_{n,j}\right)z^n \tag{1.7}$$

2. CONVOLUTION PROPERTIES

Theorem 2.1; If $f_j(z)\in TS^*(\alpha_j,\beta)$ $(j=1,....,m)$, then $(f_1*...f_m)(z)\in TS^*(\rho,\beta)$

where

$$\rho=1-\frac{(n-1)(c)_{n-1}\prod_{j=1}^{m}(1-\alpha_j)}{(a)_{n-1}\prod_{j=1}^{m}[n(1+\beta)-(\alpha_j+\beta)]-(c)_{n-1}\prod_{j=1}^{m}(1-\alpha_j)} \tag{2.1}$$

The result is sharp for the functions $f_j(z)$ given by

$$f_j(z) = z - \left(\frac{1-\alpha_j}{n(1+\beta) - (\alpha_j + \beta)} \frac{(c)_{n-1}}{(a)_{n-1}} \right) z^n \qquad 2.2$$

Proof; Following the work of Owa [1,2 (1992)], Owa and Srivastava (2003), we use the principle of mathematical induction in our proof of Theorem2.1.
Let $f_1(z) \in TS^*(\alpha_1, \beta)$ and $f_2(z) \in TS^*(\alpha_2, \beta)$. Then the inequality

$$\sum_{n=2}^{\infty} [n(1+\beta) - (\alpha_j + \beta)] \frac{(a)_{n-1}}{(c)_{n-1}} a_{n,j} \le 1 - \alpha_j \qquad (j = 1,2)$$

that is, for $m = 1$, we see that $\rho = \alpha_1$. For $m = 2$ Theorem A gives

$$\sum_{n=2}^{\infty} \sqrt{\frac{n(1+\beta) - (\alpha_j + \beta)}{1-\alpha_j} \frac{(a)_{n-1}}{(c)_{n-1}} a_{n,j}} \le 1 \qquad (j = 1,2) \qquad 2.3$$

Thus by applying the Cauchy-Schwarz inequality we have

$$\left| \sum_{n=2}^{\infty} \sqrt{\frac{[n(1+\beta) - (\alpha_1 + \beta)][n(1+\beta) - (\alpha_2 + \beta)]}{(1-\alpha_1)(1-\alpha_2)} \frac{(a)_{n-1}}{(c)_{n-1}} (a_{n,1})(a_{n,2})} \right|^2 \le$$

$$\left(\sum_{n=2}^{\infty} \frac{n(1+\beta) - (\alpha_1 + \beta)}{1-\alpha_1} a_{n,1} \right) \left(\sum_{n=2}^{\infty} \frac{n(1+\beta) - (\alpha_2 + \beta)}{1-\alpha_2} a_{n,2} \right) \frac{(a)_{n-1}}{(c)_{n-1}} \le 1$$

Therefore, if

$$\sum_{n=2}^{\infty} \frac{n-\delta}{1-\delta} (a_{n,1})(a_{n,2}) \le \sum_{n=2}^{\infty} \sqrt{\frac{[n(1+\beta) - (\alpha_1 + \beta)][n(1+\beta) - (\alpha_2 + \beta)]}{(1-\alpha_1)(1-\alpha_2)} \frac{(a)_{n-1}}{(c)_{n-1}} (a_{n,1})(a_{n,2})}$$

that is, if

$$\sqrt{(a_{n,1})(a_{n,2})} \le \frac{1-\delta}{n-\delta} \sqrt{\frac{[n(1+\beta) - (\alpha_1 + \beta)][n(1+\beta) - (\alpha_2 + \beta)]}{(1-\alpha_1)(1-\alpha_2)} \frac{(a)_{n-1}}{(c)_{n-1}}}$$

then, $(f_1 * f_2)(z) \in TS^*(\delta, \beta)$

We also note that the inequality (2.3) yields

$$\sqrt{a_{n,j}} \le \sqrt{\frac{1-\alpha_j}{n(1+\beta) - (\alpha_j + \beta)} \frac{(c)_{n-1}}{(a)_{n-1}}} \qquad (n = 2,3,\ldots \quad j = 1,2)$$

Consequently, if

$$\sqrt{\frac{(1-\alpha_1)(1-\alpha_2)}{[n(1+\beta)-(\alpha_1+\beta)][n(1+\beta)-(\alpha_2+\beta)]}\frac{(c)_{n-1}}{(a)_{n-1}}} \le \frac{1-\delta}{n-\delta}\sqrt{\frac{[n(1+\beta)-(\alpha_1+\beta)][n(1+\beta)-(\alpha_2+\beta)]}{(1-\alpha_1)(1-\alpha_2)}\frac{(a)_{n-1}}{(c)_{n-1}}}$$

That is, if

$$\frac{n-\delta}{1-\delta} \le \frac{[n(1+\beta)-(\alpha_1+\beta)][n(1+\beta)-(\alpha_2+\beta)]}{(1-\alpha_1)(1-\alpha_2)}\frac{(a)_{n-1}}{(c)_{n-1}} \qquad (n=2,3,\ldots) \tag{2.4}$$

then, we have $(f_1 * f_2)(z) \in TS^*(\delta,\beta)$. It follows from (2.4) that

$$\delta \le 1-\frac{(n-1)(c)_{n-1}[(1-\alpha_1)(1-\alpha_2)]}{[n(1+\beta)-(\alpha_1+\beta)][n(1+\beta)-(\alpha_2+\beta)](a)_{n-1}-(1-\alpha_1)(1-\alpha_2)(c)_{n-1}}$$

which shows that $(f_1 * f_2)(z) \in TS^*(\delta,\beta)$, where

$$\delta = 1-\frac{(n-1)(c)_{n-1}[(1-\alpha_1)(1-\alpha_2)]}{[n(1+\beta)-(\alpha_1+\beta)][n(1+\beta)-(\alpha_2+\beta)](a)_{n-1}-(1-\alpha_1)(1-\alpha_2)(c)_{n-1}}$$

Therefore, the result is true for $m=2$

Next we suppose that the result is true for any positive integer m. Then we have

$$(f_1 * f_2 * \ldots * f_m)(z) \in TS^*(\tau,\beta)$$

where

$$\tau = 1-\frac{(n-1)(c)_{n-1}\prod_{j=1}^{m}(1-\alpha_j)}{(a)_{n-1}\prod_{j=1}^{m}[n(1+\beta)-(\alpha_j+\beta)]-(c)_{n-1}\prod_{j=1}^{m}(1-\alpha_j)}$$

and ρ is given by (2.1). After a simple calculation we obtain

$$\rho = 1-\frac{(n-1)(c)_{n-1}\prod_{j=1}^{m+1}(1-\alpha_j)}{(a)_{n-1}\prod_{j=1}^{m+1}[n(1+\beta)-(\alpha_j+\beta)]-(c)_{n-1}\prod_{j=1}^{m+1}(1-\alpha_j)}$$

This shows that the result is true for $m+1$. Therefore, by mathematical induction the result is true for any positive integer m.

Further, taking the function $f_j(z)$ defined by (2.2) we have

$$(f_1 * \ldots * f_m)(z) = z-\left\{\left(\frac{(1-\alpha_j)}{[n(1+\beta)-(\alpha_j+\beta)]}\frac{(c)_{n-1}}{(a)_{n-1}}\right)\right\}z^n = z - A_n z^n$$

where

$$A_n = \prod_{j=1}^{m}\frac{1-\alpha_j}{[n(1+\beta)-(\alpha_j+\beta)]}\frac{(c)_{n-1}}{(a)_{n-1}}$$

It follows that

$$\sum_{n=2}^{\infty}\left(\frac{n(1+\beta)-(\rho+\beta)}{1-\rho}\frac{(a)_{n-1}}{(c)_{n-1}}\right)A_n = 1$$

This evidently complete the proof of Theorem2.1.
Letting $\alpha_j = \alpha(j = 1,\ldots,m)$ in Theorem 2.1, we have

Corollary A; If $f_j(z) \in TS^*(\alpha,\beta) \quad (j = 1,\ldots,m)$, then

$$(f_1 * f_2 * \ldots * f_m)(z) \in TS^*(\rho,\beta)$$

where

$$\rho = 1 - \frac{(n-1)(c)_{n-1}(1-\alpha)^m}{(a)_{n-1}(n(1+\beta)-(\alpha+\beta))^m - (1-\alpha)^m(c)_{n-1}}$$

The result is sharp for the functions $f_j(z) \quad (j = i,2,\ldots,m)$ given by

$$f_j(z) = z - \left(\frac{1-\alpha}{n(1+\beta)-(\alpha+\beta)}\frac{(c)_{n-1}}{(a)_{n-1}}\right)z^n \qquad (j = 1,2,\ldots,m)$$

Setting $\alpha = -1$ and $\beta = 0$ in Corollary A to obtain

Corollary B; If $f_j(z) \in TS^*(-1,0) \quad (j = 1,2,\ldots,m)$, then

$$(f_1 * f_2 * \ldots * f_m)(z) \in TS^*(\rho,0)$$

where

$$\rho = 1 - \frac{2^m(n-1)(c)_{n-1}}{(n+1)^m(a)_{n-1} - 2^m(c)_{n-1}}$$

The result is sharp for the functions $f_j(z) \quad (j = 1,2,\ldots,m)$ given by

$$f_j(z) = z - \left(\frac{2}{n+1}\frac{(c)_{n-1}}{(a)_{n-1}}\right)z^n \qquad (j = 1,2,\ldots,m)$$

Setting $\beta = 0$ in Theorem2.1, we have

Corollary C; If $f_j(z) \in TS^*(\alpha_j,0) \quad (j = 1,2,\ldots,m)$, then

$$(f_1 * f_2 * \ldots * f_m)(z) \in TS^*(\rho,0)$$

where

$$\rho = 1 - \frac{(n-1)(c)_{n-1}\prod_{j=1}^{m}(1-\alpha_j)}{(a)_{n-1}\prod_{j=1}^{m}(n-\alpha_j) - (c)_{n-1}\prod_{j=1}^{m}(1-\alpha_j)}.$$

The result is sharp for the functions

$$f_j(z) = z - \left(\frac{1-\alpha_j}{n-\alpha_j} \frac{(c)_{n-1}}{(a)_{n-1}} \right) z^n \qquad (j = 1,2,\dots,m)$$

By fixing the second Coefficient, and putting $\beta = 0$ we have the following

Corollary D; If $f_j(z) \in TS^*(\alpha_j, \beta) \quad (j = 1,2,\dots,m)$

$$(f_1 * f_2 * \dots * f_m)(z) \in TS^*(\rho, \beta)$$

where

$$\rho = 1 - \frac{c \prod_{j=1}^{m} (1-\alpha_j)}{a \prod_{j=1}^{m} (2-\alpha_j) - c \prod_{j=1}^{m} (1-\alpha_j)}$$

The result is sharp for the functions $f_j(z) \quad (j = 1,2,\dots,m)$ given by

$$f_j(z) = z - \left(\frac{c(1-\alpha_j)}{a(2-\alpha_j)} \right) z^2 \qquad (j = 1,2,\dots,m)$$

After fixing second coefficient as in Theorem B, we have the next Theorem

Theorem 2.2; If $f_j(z) \in TS_b^*(\alpha_j, \beta) \quad (j = 1,2,\dots,m)$, then

$$(f_1 * \dots * f_m)(z) \in TS_b^*(\rho, \beta)$$

where

$$\rho = 1 - \frac{(n-1)(1-b)(c)_{n-1} \prod_{j=1}^{m} (1-\alpha_j)}{(a)_{n-1} \prod_{j=1}^{m} [n(1+\beta) - (\alpha_j + \beta)] - (1-b)(c)_{n-1} \prod_{j=1}^{m} (1-\alpha_j)} \qquad 2.5$$

The result is sharp for the functions $f_j(z) \quad (j = 1,2,\dots,m)$, given by

$$f_j(z) = z - \left(\frac{(1-b)(1-\alpha_j)(c)_{n-1}}{n(1+\beta) - (\alpha_j + \beta)} \frac{(c)_{n-1}}{(a)_{n-1}} \right) z^n \qquad (j = 1,2,\dots,m) \qquad 2.6$$

Proof: Following the same method as in Theorem 2.1 with some simple calculation the result follows.

Letting $\alpha_j = \alpha (j = 1,2,\dots,m)$ in Theorem 2.2; we have

Corollary E; If $f_j(z) \in TS_b^*(\alpha, \beta) \quad (j = 1,2,\dots,m)$, then

$$(f_1 * \dots * f_m)(z) \in TS_b^*(\rho, \beta)$$

where

$$\rho = 1 - \frac{(n-1)(1-b)(c)_{n-1}(1-\alpha)^m}{(a)_{n-1}[n(1+\beta)-(\alpha+\beta)]^m - (1-b)(1-\alpha)^m(c)_{n-1}}$$

The result is sharp for the functions $f_j(z)$ $(j = 1,2,\ldots,m)$ given by

$$f_j(z) = z - \left(\frac{(1-b)(1-\alpha)(c)_{n-1}}{n(1+\beta)-(\alpha+\beta)(a)_{n-1}}\right)z^n \qquad (j = 1,2,\ldots,m)$$

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