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EFFECT OF COOLING ON THE ONSET OF THERMOSOLUTAL INSTABILITY OF A FLUID LAYER HEATED FROM BELOW WITH FIRST-ORDER CHEMIC AL REACTION IN A POROUS MEDIUM

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ABSTRACT

This paper investigates the effect of cooling on the onset of thermosolutal instability in a horizontal fluid layer filled with binary fluid with first-order chemical reaction in a porous medium. The cooling is taken to be of Newton's type; while the linear stability analysis is employed to determine the onset of the buoyancy-driven convective motion. It is seen that the effect of cooling cannot be ignored both for the static temperature and on the propagation of the disturbances in the system. In addition, cooling delayed the onset of instability and higher values led to greater stabilization; while the presence of chemical reaction led to destabilization of the system.

KEYWORDS: Thermosolutal instability, Newton's cooling, binary fluid layer, porous medium, chemical reaction.

1. INTRODUCTION

Combined natural heat and mass transfer, the so-called double-diffusive or thermosolutal convection, where the flows are induced by both temperature and solute fields has received considerable attention due to its important applications in diverse fields such as in the dispersal of chemical contaminants through water saturated soil, the exploitation of continental geothermal reservoir, metallurgy, geophysics, enhanced oil recovery, and packed-bed catalytic reactors, food processing and the migration of moisture in fibrous insulation, (Bahloul *et. al* (2003); Hill (2005)). Also, it is likely to prove essential in the understanding of some areas of stellar convection (Huppert (1977)). Thermosolutal convection owes its existence to the presence of two components of different molecular diffusivities which contribute in an opposing sense to the locally vertical density gradient. That is, thermosolutal convection is due to density variations which are induced by both temperature and concentration gradients where the heat and solute concentration are transported by convection and diffusion (Bahloul *et. al.*, (2003)).

Early investigations on double-diffusive natural convection in porous media focused on the problem of convective instability analysis in a horizontal layer. For example Nield (1968); Taslim and Narusawa (1986); Brand and Steinberg (1983); Malashtly (1994) used linear stability analysis to determine the criteria for the onset of thermohaline convection for various conditions; while Rudraiah *et. al.* (1982a, b) investigated the region of instability via salt – finger and diffusive regimes in a porous layer for thermohaline convection. More recently, Lombardo *et. al.* (2001) studied linear and non – linear stability of a Bernard problem for double – diffusive mixture in a fluid – saturated porous medium using the direct method of Lyapunov.

However, in all these studies the effects of cooling and chemical reaction have been ignored. In systems in which the specie concentration dissociates with attendant heat generation, cooling effect becomes significant. The aim of this paper therefore, is to investigate the effect of cooling on the onset of thermosolutal instability in a horizontal fluid layer filled with binary fluid with first-order chemical reaction in a porous medium using linear stability analysis technique.

2 MATHEMATICAL FORMULATION

We consider a horizontal double-diffusive fluid layer of height $d > 0$ in a porous medium heated from below and confined between two parallel surfaces located at $z' = \frac{-d}{2}$ and $z' = \frac{d}{2}$. The schematic diagram of the physical model and coordinate system is shown in Fig. 1. Initially the fluid is at rest when the lower surface is maintained at temperature $T_1 = T_0 + \frac{\Delta T}{2}$ and concentration $c_1 = c_0 + \frac{\Delta c}{2}$; while the upper surface is at temperature $T_2 = T_0 - \frac{\Delta T}{2}$ and concentration $c_2 = c_0 - \frac{\Delta C}{2}$. Here $T_0 = \frac{(T_1 + T_2)}{2}$, $c_0 = \frac{(c_1 + c_2)}{2}$, $T_1 > T_2$, $c_1 > c_2$.

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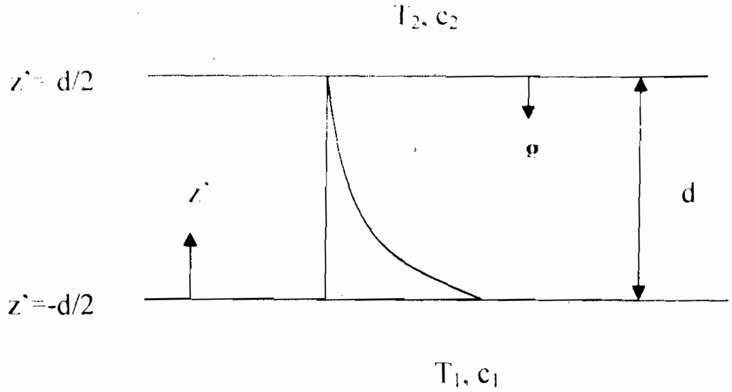

Fig. 1. Schematic diagram of the physical model and Coordinate system

The porous medium is assumed to compose of sparse distributed particles, while the binary fluid is assumed to be incompressible and to satisfy the Boussinesq approximation. The density variation and concentration is described by (Lombardo *et. al.* (2001); Lawson and Yang (1973))

$$\rho = \rho_0\left[1 - \beta_T(T' - T_0) + \beta_c(C' - C_0)\right] \tag{1}$$

where ρ_0 is the density of the fluid mixture at temperature $T' = T_0$ and mass fraction $C' = C_0$ and β_T and β_c are the thermal and concentration expansion coefficients, respectively. The subscript "0" refers to condition at the origin of the coordinate system.

The governing equations of the motion for the flow of a binary mixture through a porous medium under the usual Boussinesq approximation taking into account the effects of cooling and first-order chemical reaction are (Chhuon and Caltagirone (1979); Phillips (1991))

$$\nabla' \bullet \mathbf{v}' = 0 \tag{2}$$

$$\frac{\partial \mathbf{v}'}{\partial t'} + (\mathbf{v}' \bullet \nabla')\mathbf{v}' = -\frac{1}{\rho_0}\nabla' P' - g\beta_T(T' - T_0)\vec{k} + g\beta_c(C' - C_0)\vec{k} - \frac{\nu}{k}\mathbf{v}' + \nu\nabla'^2\mathbf{v}' \tag{3}$$

$$\frac{1}{M}\frac{\partial T'}{\partial t'} + (\mathbf{v}' \bullet \nabla')T' = \frac{\kappa}{(\rho_0 c_p)_f}\nabla'^2 T' - \frac{\gamma^2}{(\rho_0 c_p)_f}(T' - T_0) \tag{4}$$

$$\phi\frac{\partial c'}{\partial t'} + (\mathbf{v}' \bullet \nabla')C' = D\nabla'^2 C' - k_r'^2(C' - C_0) \tag{5}$$

where $\mathbf{v}'$ is the velocity vector, P' the pressure, g is the acceleration due to gravity, $\nu = \frac{\mu}{\rho_0}$ is the kinematic viscosity, κ the thermal diffusivity, D the diffusion coefficient, ρ_0 the density, c_p is the specific heat capacity of the fluid, $k_r'^2$ is the reaction coefficient and γ^2 is the cooling coefficient. Also, the parameter ϕ is the normalized porosity defined by

$$\phi = \phi' M \tag{6}$$

where ϕ' is the porosity and M is the ratio of the heat capacities defined by

$$M = \frac{(\rho_0 c_p)_f}{(\rho_0 c_p)_m} \tag{7}$$

In Eq. (7) the subscripts "*f*" and "*m*" refer to the fluid and medium respectively.

The boundary conditions are (Lombardo *et. al.* (2001))

$$w' = 0, T' = T_1, C' = c_1 \quad \text{at} \quad z' = \frac{-d}{2}$$
$$w' = 0, T' = T_2, C' = c_2 \quad \text{at} \quad z' = \frac{d}{2} \tag{8}$$

The non-dimensional equations representing the flow are governed by

$$\nabla \bullet \mathbf{V} = 0 \tag{9}$$

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \bullet \nabla)\mathbf{V} = -\nabla P - \frac{1}{\Pr}(R_T T - R_S C)\vec{k} - \chi^2 \mathbf{V} + \nabla^2 \mathbf{V} \tag{10}$$

$$\Pr\left(\sigma \frac{\partial T}{\partial t} + (\mathbf{V} \bullet \nabla)T\right) = \left(\nabla^2 - B^2\right)T \tag{11}$$

$$Sc\left(\phi \frac{\partial C}{\partial t} + (\mathbf{V} \bullet \nabla)C\right) = \left(\nabla^2 - R^2\right)C \tag{12}$$

subject to the conditions

$$W = 0,\ T = \pm\frac{1}{2},\ C = \pm\frac{1}{2} \quad \text{at} \quad z = \mp\frac{1}{2} \tag{13}$$

where we have used in Eqs.(9-13) the following non-dimensional variables and parameters

$$t = \frac{t'\nu}{d^2},\ \mathbf{V} = \frac{d}{\nu}\mathbf{V}',\ \nabla = d\nabla',\ (x,y,z) = \frac{1}{d}(x',y',z'),\ T = \frac{T' - T_0}{T_1 - T_2},\ C = \frac{C' - C_0}{c_1 - c_2},$$

$$\Pr = \frac{\nu}{\alpha},\ \alpha = \frac{\kappa}{(\rho_0 c_p)_f},\ R^2 = \frac{d^2 k_r'^2}{D},\ B^2 = \frac{d^2\gamma^2}{\kappa},\ Sc = \frac{\nu}{D},\ P = \frac{d^3}{\rho_0 \nu^2}P',\ \chi^2 = \frac{d^2}{k}$$

$$R_T = \frac{g\beta_T d^3 (T_1 - T_2)}{\alpha\nu},\ R_C = \frac{g\beta_C (c_1 - c_2)}{\alpha\nu},\ \sigma = \frac{1}{M} \tag{14}$$

Here, R_T is the thermal Rayleigh number and R_C the solutal Rayleigh number.

3.1 Basic Flow

The basic state of the system is given by the static solution $\mathbf{V} = 0$ and $\frac{\partial}{\partial t} \to 0$ of the governing equations (9)-(12) subject to boundary conditions (13) to which corresponds the static temperature, T_S; static concentration, C_S and static pressure P_S given respectively by

$$\left(D^2 - B^2\right)T_S = 0 \tag{15}$$

$$\left(D^2 - R^2\right)C_S = 0 \tag{16}$$

$$DP_S = -\frac{1}{\Pr}\left(R_T T_S - R_C C_S\right) \tag{17}$$

where $D \equiv \frac{d}{dz}$. Equations (15) – (17) are to be solved subject to the conditions

$$T_S = \pm\frac{1}{2},\ C_S = \pm\frac{1}{2} \qquad \text{at } z = \mp\frac{1}{2} \tag{18}$$

Solving equations (15) - (17) using boundary conditions (18) yield

$$T_S = \frac{1}{2}\frac{Sinh(Bz)}{Sinh(R/2)} \tag{19}$$

$$C_S = \frac{1}{2}\frac{Sinh(Rz)}{Sinh(R/2)} \tag{20}$$

$$P_S = -\frac{1}{\Pr}\int\left(R_T T_S - R_C C_S\right)dz \tag{21}$$

3.2 Effect of perturbation (Linearization) and normal mode analysis

In order to study the stability of the static state and thus determine the criterion for the onset of the Rayleigh-Bernard instability, we consider the following perturbed state defined by (Chandrasekhar (1961); Drazin and Reid (2004))

$$\mathbf{V} = \mathbf{0} + \mathbf{v},\ T = T_S + \theta,\ C = C_S + \varphi,\ P = P_S + p \tag{22}$$

Following the classical procedure for linear mode analysis of Chandrasekhar (1961), we substitute Eq. (22) into Eqs. (9)- (12) and the boundary condition (13) to obtain the following linearize equations

$$\nabla \bullet \mathbf{v} = 0 \tag{23}$$

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla p - \frac{1}{\Pr}(R_T\theta - R_S\varphi)\vec{k} - \chi^2\mathbf{v} + \nabla^2\mathbf{v} \tag{24}$$

$$\Pr\left(\sigma\frac{\partial \theta}{\partial t} - \alpha_1 w\right) = \left(\nabla^2 - B^2\right)\theta \tag{25}$$

$$Sc\left(\phi\frac{\partial \varphi}{\partial t} - \alpha_2 w\right) = \left(\nabla^2 - R^2\right)\varphi \tag{26}$$

$$w = 0 = \theta = \varphi \quad \text{at} \quad z = \pm\frac{1}{2} \tag{27}$$

where $\alpha_1 = -\frac{\partial T_S}{\partial z}$ and $\alpha_2 = -\frac{\partial C_S}{\partial z}$ are the non-dimensional static temperature and concentration gradients, respectively. Next we take the double curl of Eq. (24), using Eq. (23) and keeping only the vertical component of the velocity, w to obtain

$$\left(\frac{\partial}{\partial t} + \chi^2 - \nabla^2\right)\nabla^2 w = -\frac{1}{\Pr}R_T\nabla_h^2\theta + \frac{1}{\Pr}R_T\nabla_h^2\varphi \tag{28}$$

where $\nabla_h^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

We now study the of the normal mode disturbances which are chosen in our case to be two dimensional waves in the horizontal plane (x, y). To this end we search for solutions of the form (Chandrasekhar (1961))

$$\begin{aligned} w(x,y,z,t) &= W(z)Exp[i(a_x x + a_y y) + \Omega t] \\ \theta(x,y,z,t) &= \Theta(z)Exp[i(a_x x + a_y y) + \Omega t] \\ \varphi(x,y,z,t) &= \Psi(z)Exp[i(a_x x + a_y y) + \Omega t] \end{aligned} \tag{29}$$

where $\Omega = \Omega_R + i\Omega_I$ is a complex number and Ω_R, Ω_I are real numbers. Substituting Eqs. (29) into (25), (26), (28) and the boundary conditions (27) yield

$$(D^2 - k^2 - B^2 - \Pr\sigma\Omega)\Theta = -\Pr\alpha_1 W \tag{30}$$

$$(D^2 - k^2 - R^2 - Sc\phi\Omega)\Psi = -Sc\alpha_2 W \tag{31}$$

$$(D^2 - k^2 - \chi^2 - \Omega)(D^2 - k^2)W = -\frac{1}{\Pr}R_T k^2\Theta + \frac{1}{\Pr}R_S k^2\Psi \tag{32}$$

$$W = 0 = \Theta = \Psi \quad \text{at} \quad z = \pm\frac{1}{2} \tag{33}$$

$$D^2 W = 0 \text{ on a free surface} \tag{34}$$

where $D \equiv \frac{\partial}{\partial z}$ and $k = a_x^2 + a_y^2$.

Next, we study the stability of all possible disturbances for all wave numbers from the system (31) - (34). To this end, we reduce the system (31) - (34) into a single scalar equation by eliminating Θ and Ψ from the system. This is achieved by operating on Eq. (32) with $(D^2 - k^2 - \Pr M\Omega)(D^2 - k^2 - R^2 - Sc\phi\Omega)$ and using (30) and (31) to obtain

$$\begin{aligned} &((D^2 - k^2)(D^2 - k^2 - \chi^2 - \Omega)(D^2 - k^2 - B^2 - \Pr M\Omega)(D^2 - k^2 - R^2 - Sc\phi\Omega) \\ &- R_T k^2\alpha_1(D^2 - k^2 - R^2 - Sc\phi\Omega) + \frac{Sc}{\Pr}R_S\alpha_2 k^2(D^2 - k^2 - B^2 - \Pr M\Omega))W = 0 \end{aligned} \tag{35}$$

Now, for idealized free-free boundaries Eq. (35) has solution of the form (Chandrasekhar (1961))

$$W = w_0 Sin\pi z \tag{36}$$

where w_0 is a constant. Substituting Eq.(36) into Eq.(35) and simplifying yields the Rayleigh number as

$$\begin{aligned} R_T = \frac{1}{k^2(\pi^2 + k^2 + R^2)}\Big\{&(\pi^2 + k^2)(\pi^2 + k^2 + \chi^2 + \Omega)(\pi^2 + k^2 + R^2 + \Omega Sc\phi) \\ &ScR_S k^2\Gamma_1(\pi^2 + k^2 + B^2 + \Omega\Pr M)\Big\}\Gamma_2 \end{aligned} \tag{37}$$

where $\Gamma_1 = \dfrac{RCosh(Rz)}{2\Pr Sinh(R/2)}$, $\Gamma_2 = \dfrac{2Sinh(B/2)}{BCosh(Bz)}$.

For the analysis of stationary convection, that is the onset of instability, we put $\Omega = 0, R_T = Ras$ in Eq. (37) and obtain

$$Ras = \frac{1}{k^2(\pi^2+k^2+R^2)}\Big\{(\pi^2+k^2)(\pi^2+k^2+\chi^2)(\pi^2+k^2+R^2)(\pi^2+k^2+B^2) + ScR_S k^2\Gamma_1(\pi^2+k^2+B^2)\Big\}\Gamma_2 \tag{38}$$

The critical value of the wave number $k = k_c$ is determined by finding the minimum of $Ras(k_c)$ and following Chandrasekhar (1961) we minimize Eq.(38) as follows

$$\frac{\partial Ras(k_c)}{\partial k_c^2} = 2k_c^{10} + a_8k_c^8 + a_6k_c^6 + a_4k_c^4 - a_2k_c^2 - a_0 = 0 \tag{39}$$

where

$$a_0 = B^2\pi^2(\pi^2+R^2)(\pi^2+\chi^2)$$

$$a_2 = [2\pi^4 + \pi^4(\pi^2+R^2) + 2B^2\pi^2](\pi^2+R^2)(\pi^2+\chi^2)$$

$$a_4 = 2\pi^6 + 6\pi^4R^2 + R^2R_SSc\Gamma_1 + R^4\chi^2 + \pi^2R^2(3R^2+2\chi^2) + R^4 - R_SSc\Gamma_1 + \pi^2(2R^2-\chi^2)$$

$$a_6 = 2(\pi^2+R^2)(4\pi^2+R^2+B^2+\chi^2)$$

$$a_8 = 7\pi^2 + 4R^2 + B^2 + \chi^2$$

From Eq.(38), the critical Rayleigh number, Rac is obtained by substituting $k = k_c$, that is

$$Rac = \frac{\Gamma_2}{k_c^2(\pi^2+k_c^2+R^2)}\Big((\pi^2+k_c^2)(\pi^2+k_c^2+\chi^2)(\pi^2+k_c^2+R^2)(\pi^2+k_c^2+B^2) + ScR_Sk_c^2(\pi^2+k_c^2+B^2)\Gamma_1\Big) \tag{40}$$

4. RESULTS AND DISCUSSION

For the analysis of the effect of cooling on the stationary convection of thermosolutal instability of a fluid layer heated from below with first-order chemical reaction in a porous medium, it is observed from Eq. (40) that Rac has a minimum value when the denominator is a maximum. This value occurs at $z = 0$, that is, at the centre of the channel. Thus

$$Rac = \frac{\frac{Sinh(B/2)}{(B/2)}}{k_c^2(\pi^2+k_c^2+R^2)}\Big((\pi^2+k_c^2)(\pi^2+k_c^2+\chi^2)(\pi^2+k_c^2+R^2)(\pi^2+k_c^2+B^2) + ScR_Sk_c^2(\pi^2+k_c^2+B^2)\frac{R/2}{\Pr Sinh(R/2)}\Big) \tag{41}$$

Also, Eq. (39) is a tenth degree polynomial in the critical wave number, k_c and its roots cannot be easily represented analytically. So we seek for numerical solution of k_c from which we obtain the critical Rayleigh number for different values of the parameters entering into Eq. (39). Solving Eq. (39) using the software "mathematica" (Wolfram (1991)) and the following values for salt water ($R_S = 1000, \Pr = 7, Sc = 700$) and $\chi = 0.5$, $R = 0.2$, $B = 0.1$, we obtain the critical wave number as $k_c = 2.22$. However, in the absence of cooling, B, chemical reaction, R and porosity, χ the critical Rayleigh number is $\frac{27}{4}\pi^4 + \frac{ScR_S}{\Pr}$ with the critical value wave number of $(\frac{\pi}{\sqrt{2}})$. Further in the absence of the solutal Rayleigh number this result is in good agreement with the critical value for the classical Rayleigh-Bernard problem of Chandrasekhar (1961) where $Rac = \frac{27}{4}\pi^4$.

Next, we investigate the numerical behaviour of the effect of cooling parameter, *B* on the critical Rayleigh number given in Eq (41) with $k_c = 2.22$ and various values of the parameters χ, R, $R_S, Sc, \Pr$. The results are shown in Table 1.

Table 1: Effect of cooling on the critical Rayleigh number, Rac for $Sc = 700$, $R_S = 1000$, $\Pr = 7$ and various values of χ, R.

k_c	B	Rac $\chi = 0.5, R = 0.2$	Rac $\chi = 0, R = 0$
2.22	0.0	100233	100491
2.22	0.1	100343	100601
2.22	0.5	102991	103257
2.22	1.0	111521	111808
2.22	1.5	126607	126933

From table 1, it is observed that increase in the cooling parameter, *B* delays the onset of instability; while the presence of reaction enhances the onset of instability in the system. In addition, greater cooling is associated with greater stabilization whereas the reverse is the case with higher values of the reaction parameter.

5. CONCLUSIONS

Although many studies already exist on the stability analysis of double diffusive convection in a horizontal fluid layer heated from below in a porous medium, none has considered the combined effect of cooling and first-order chemical reaction. In this paper we have studied the combined effects of cooling and first-order chemical reaction in a horizontal fluid layer heated from below in a porous medium.

Equation (37) when $\Omega = 0$ gives the value $R_T = Ras$ for stationary convection given by Eq. (38); Eq. (41) gives the critical value of the Rayleigh number Rac for $k = k_c$ which corresponds to the minimum of $Ras(k_c)$ at the centre of the horizontal channel $(z = 0)$ and the onset of stationary convection at pitchfork bifurcation. Thus, the main results of this paper using salt water parameters are

- cooling delays the onset of instability
- the presence of reaction enhances the onset of instability in the system
- greater cooling is associated with greater stabilization
- higher values of the reaction parameter results in greater destabilization of the system.

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