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CYCLICAL SUBNORMAL SEPARATED A-GROUPS OF NILPOTENT LENGTH N

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ABSTRACT

The existence of Cyclically Subnormally Separated A-groups of nilpotent length n where n is any positive integer has been proved. This shows that there are A-groups of any nilpotent length n in CSn.

KEYWORDS: Cyclical Subnormal group, A-Groups of Nilpotent length n.

1. INTRODUCTION

In Markafi, (2006 and 2005) the existence of CSn, A-groups of nilpotent length four and of nilpotent lengths five and six were presented. The purpose of this communication is to show that there are A-groups of any nilpotent length n in CSn, where n is a positive integer.

A sketch of the case for nilpotent length seven is given before plunging into the general case. The schematic diagram (Figure 1) and the ones given in the two papers sighted above bring out in a simple form the beauty of the subgroup structure in these groups.

2. NILPOTENT LENGTH SEVEN

This case is presented in a way similar to that of nilpotent length six. The diagram (Fig. 1) is to help the mind to imagine what is going on as we go through the various subgroups in the general discussion. The notation is the same as the one adopted in the nilpotent length six and it is easy to see that the group presented is in fact a CSn A-group of nilpotent length seven.

$G = P_7 \rtimes H$ nilpotent length seven

P_1 and P_2 cyclic of orders P_1^6 and P_2^6 respectively. P_3, P_4 and P_5 homocyclic of exponents P_3^4, P_4^3 and P_5^2 respectively. P_6 and P_7 elementary abelian P_6 and P_7–groups.

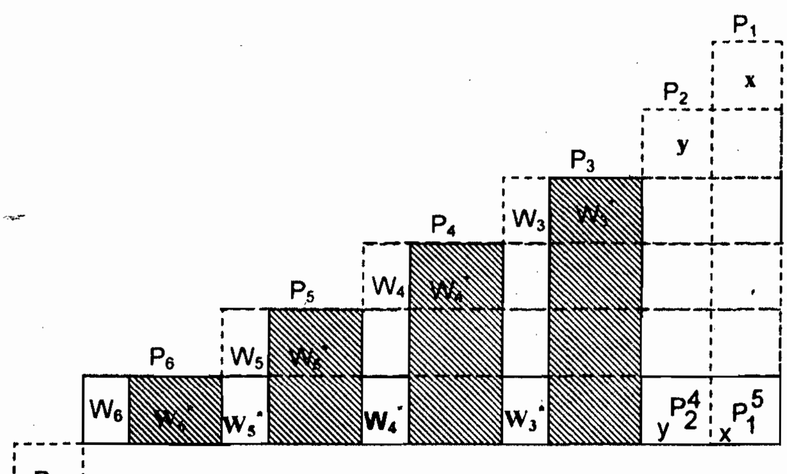

Figure 1

$k = \langle w_6 \otimes 1, w_6^{*}, w_5^{*}, w_5^{*}, w_4^{*}, w_4^{*}, w_3^{*}, w_3^{*}, w_3^{*}, w_3^{*}, y^{p_2^4}, x^{p_1^5} \rangle$
$L = \langle w_6^{*}, w_5^{*}, w_4^{*}, w_3^{*} \rangle$

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3. NILPOTENT LENGTH n

We start the general construction by recalling some of the facts that will guide us in the various steps we shall take. We know from theorem (2.2) (Markafi, 2006) that an A-group in CSn all of whose Sylow p-subgroups have exponent at most p^{n-1} will have nilpotent length at most n. We also know that the *nilpotent length* of any A-group cannot exceed the number of the distinct primes that divide its order, theorem (8.3) of (Taunt, 1949). So we know that an A-group in CS_n whose order is divisible by n distinct primes and whose Sylow p-subgroups are of exponent at most p^{n-1} will have nilpotent length at most n. This and theorem (2.7) (Markafi 2006) gave us an idea of the number of abelian p-groups we shall need and their exponents.

As we go through the construction we choose our primes in the following way. Suppose we construct the H_r a group of nilpotent length r, we choose the next prime P_{r+1} such that the order of H_r divides $P_{r+1}-1$ where

$$H_r = P_r \rtimes H_{r-1}; \qquad 1 \le r \le n-1$$

with $H_0 = 1$ and $H_1 = P_1$. We let $P_1 = <x>$ and $P_2 = <y>$ be cyclic of orders P_1^{n-1} and P_2^{n-1} respectively. The next step is to form

$$H_2 = P_2 \rtimes P_1$$

in a way similar to the H_2 in the construction for the nilpotent five case (Makarfi 2005)

For the rest of the prime power order p-groups we get
P_r, $3 \le r \le n-1$

in the following way. P_r is a homocyclic p_r-groups of exponent m and is the result of inducing up to H_{r-1}, a rank one K_{r-1} –module, $W_r = <w_r>$, over the ring $k_r = Z/mZ$; where

$$m = p_r^{n-r} \text{ and } K_{r-1} = < w_{r-1}, W^*_{r-1}, w^*_{r-2}, W^*_{r-2}, \dots, w^*_3, W^*_3, w^*_2, w^*_1 >.$$

Here

$$w^*_i = (w_i \otimes 1)k, \quad k \cong P_j^{r-i}, \; 3 \le i \le r-1; \; w^*_2 = y^{P_2^{r-3}}, \; w^*_1 = x^{P_1^{r-2}}$$

and

$$W^*_i = < w_i \otimes t : t \in T^*_{i-1}; \; 3 \le i \le r-i>$$

where T_{i-1} is a transversal to K_{i-1} in H_{i-1} and $T^*_{i-1} = T_{i-1} \backslash \{1\}$.
So that W_r^* is K_{r-1} in variant. We then have

$$P_r = < w_r \otimes 1, \; W_r^* >; \; 3 \le r \le n-1$$

To see what is going on, we note that if r = 3.

$$K_2 = < w^*_2; \; w^*_1 > = < y, \; x^{P_1} >$$

while if r = n-1

$$K_{n-2} = < w^*_{n-2}, W^*_{n-2}, \dots, w^*_3, W^*_3, \; y^{P_2^{n-4}}, x^{P_1^{n-3}} >.$$

We next get P_n which is elementary abelian and is the result of a similar inducing process as above from

$$K_{n-1} = < w_l^*, W_l^*; \; 3 \le l \le n-1; \; y^{P_2^{n-3}}, \; x^{P_1^{n-2}} >$$

We construct the group by induction on its nilpotent length n, so we assume we have an A-group H_{r-1} in CSn of nilpotent length r-1 and we have

$L_{r-1} \lhd K_{r-1}$ with K_{r-1}/L_{r-1} cyclic of order $p_1 p_2 \dots p_{r-1}$ where K_{r-1} is as given above and

$$L_{r-1} = < w_l^{*P_l} w_l^* P_l, W_j^* \mid 1 \le l \le r-1, \; 3 \le j \le r-1 >$$

The action of K_{r-1}/L_{r-1} on W_r is lifted to K_{r-1}, so that K_{r-1} acts as follows,

$$w_r w^*_i = \lambda_i w_r; \quad 1 \le i \le r-1$$

and W_j^* acts trivially on w_r for $3 \le j \le r-1$. Each λ_i is an element, in k_r^* the group of units of the ring k_r, of order p_i. Then we get

$$Pr = W_r^{Hr-1} \text{ and form } H_r = P_r \rtimes H_{r-1}.$$

The next step is to define L_r and K_r and show that $L_r \lhd K_r$ and that K_r/L_r is cyclic of order $p_1 \ldots p_r$. In order to complete the induction process we also have to show that H_r is a CS_n group. Now we know that P_r is a homocyclic p_r-group of exponent p_r^{n-r} so let

$$K_r = \langle w_i^*, W_j^* ; 1 \le i \le r, 3 \le j \le r \rangle$$

where

$$w_i^* = (w_i \otimes 1)^{p_i^{r-1}} ; 3 \le i \le r, \; w_2^* = y^{p_2^{r-2}}, \; w_1^* = x^{p_1^{r-1}}$$

and

$$W_i^* = \langle w_i \otimes t | t \in T_{i-1} \backslash \{1\}, \; 3 \le i \le r \rangle$$

for some transversal T_{i-r} to K_{i-1} in H_{i-1}

$$L_r = \langle (W_i^*)_{p_i}, \; W_j | \; 1 \le l \le r, 3 \le j \le r \rangle$$

Now we note that

$$K_r = \langle \Omega_{n-r}(P_i), \; 1 \le i \le r, L_r \rangle$$

while

$$L_r = \langle \Omega_{n-r-1}(P_i), \; 1 \le i \le r, L_r \cap L_{r-1} \rangle$$

But

$$\Omega_{n-r}(P_i) \subset \Omega_{n-r+1}(P_i) \subseteq C_{H_{r-1}}(L_{r-1}); \; 1 \le i \le r-1.$$

So

$$L_r \lhd K_r$$

Clearly K_r/L_r is cyclic of order $p \ldots p_r$ generated by

$$v = \overline{w}_1^* \ldots \overline{w}_r^* \text{ where } \overline{w}_i^* = \overline{w}_i^* L_r ; 1 \le i \le r$$

In order to show that H_r is a CSn group we start by showing that L_r is subnormal in H_r. Here we observe that

$$K_r = P_r L_{r-1}$$

and because L_{r-1} is subnormal in H_{r-1} we have

$$K_r/P_r \approx L_{r-1} \text{ sn} H_{r-1} \approx H_r/P_r$$

$\therefore \quad K_r/P_r$ sn H_r/P_r

$\therefore \quad K_r$ sn H_r

Since L_r is subnormal in K_r we also have

L_r sn H_r

So using proposition (2.2), (Makarfi 2005) it only remains to show that for any element h in H_r of prime power order

$$\langle h \rangle \cap K_r = \langle h \rangle \cap L_r \Rightarrow h \in L_r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (*)$$

Now, for each $i \in \{1, 2, \ldots, r\}$

$$P_i = \langle w_i \otimes 1 \rangle \times W_i^*$$

So we have

$$K_r \cap P_i = \langle w_i^* \rangle \times W_i^* \text{ and } L_r \cap P_i = \langle (W_i^*)^{P_i} \rangle \times W_i^*$$

Now suppose that h is a p_i–element satisfying the hypothesis in (*) and Q is a Sylow p_i–subgroup of H_r containing h. Then

$Q \cap L_r$ is a Sylow p_i-subgroup of L_r because L_r is subnormal in H_r so

$$Q \cap L_r \approx P_i \cap L_r$$

Thus $Q \cap L_r$ contains a subgroup isomorphic to W_i^* say Q_o and this is a direct factor of Q because both Q and W^* are homocyclic of the same exponent. So

$$Q = Q_o \times \langle u \rangle$$

$$\therefore Q \cap L_r = Q_o \times \langle u_1 \rangle, \text{ where } \langle u_1 \rangle = \langle u \rangle \cap L_r$$

Also $Q \cap K_r$ is a Sylow p_i-subgroup of K_r and

$$Q \cap L_r \subset Q \cap K_r$$

$$\therefore \quad Q \cap K_r = Q_o \times \langle u_2 \rangle, \text{ where } \langle u_2 \rangle = K_r \cap \langle u \rangle.$$

So that $u_2^{p_i} = u_1$. If

$$h \notin Q \cap L_r$$

then $h = h_1u'$ for some $h_1 \in Q_o$ and $u' \in < u >\backslash< u_1>$ so that $< u' >$ contains $<u_2>$. Therefore $< h>$ contains h_1u_2 while $h_1u_2 \notin L_r$

and this contradicts the assumption that h satisfies the hypothesis in (*). So we conclude that $h \in L_r$. This establishes (*) and we must have H_r in CSn.

We finally conclude, by induction, that our group is a CSn group,

$G = P_n \rtimes H_{n-1}$

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