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CYCLICAL SUBNORMAL SEPARATION IN A-GROUPS

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ABSTRACT

It is shown that any A-group that is generated by elements of prime order and satisfying the cyclical subnormal separation condition is metabelian. The two other main results give necessary and sufficient conditions for A-groups, that are split extensions of certain abelian p-groups by a metabelian p'-group, to satisfy the cyclic al subnormal separation condition. There is also a result which shows that A-groups with elementary abelian Sylow subgroups are cyclically separated.

KEYWORD: A-groups, Cyclical Subnormal Separation, CS_n-groups.

1 INTRODUCTION

A-group G is called a CS_n group if for any given cyclic subgroup $B \leq G$ and a subgroup A of B, there exists a subnormal subgroup N of G such that $N \cap B = A$. This is the cyclical subnormal separation condition.

The class of finite CS_n groups was introduced in Makarfi (1991) and it was shown that this class contains the class *of finite CS groups introduced in (Hardley, et al, 1982). Also in Makarfi (1997) certain results about A-groups* were brought up. The discussions in Makarfi (1997) were clearly motivated by the appearance of the class of CS_n groups. This paper is another step in the direction of the concluding remarks on Makarfi (1997).

Three main results are presented in Section 3. The general aim is an effort to get hold of those A-groups that are in CS_n.

We also come across this result, Lemma (3.3) which concerns CS groups. We find it appropriate to bring it here because the CS-paper introduced in (Hardley, et al, 1982) was the motivating factor for the appearance of the CS_n groups. There was the general feeling that A-groups may be CS groups. Lemma (3.3) gave one positive result. But as it turns out we have an example of an A-group which is even a CS_n group but not a CS-group.

Section 2 deals with general results that are of interest to us including an example of a non-metabelian A-group in CS_n. In fact as it has been pointed out in the concluding remarks, a slight modification of this example gives an example of an A-group in CS_n which is not a CS group.

2. PRELIMINARIES

2.1 Theorem A established that any A-group in CS_n which is generated by elements of prime order must be metabelian. We begin our discussions with an example to show, first, that there are non-metabelian A-groups in CS_n and secondly that the hypothesis, in the theorem, of the generating elements to be of prime order is necessary. We note that metabelian groups had already been shown to belong to the class of CS_n groups [(3.5) of Makarfi, 1991].

2.2 Example: Suppose that p,q and r are three distinct primes and $\langle x \rangle$ is a cyclic group of order r^2. Let k be the order of q mod r, i.e. $r | q^k - 1$ and $r \nmid q^t - 1$ for $t < k$. Then the field $M = \mathbb{F}_{q^k}$ contains a primitive r^{th} root of unity say θ. So that M can be considered as an $\langle x \rangle$-module as follows:

$$\forall m \in M \text{ let } mx = \theta m$$

Then $mx^r = \theta^r m = m$, so that x^r acts trivially on M. Now form the semidirect product $M\langle x \rangle$ and let

$$H = M\langle x^r \rangle = M \times \langle x^r \rangle = M_1 \times M_2 \times \ldots \times M_k \times \langle x^r \rangle$$

where $M_i = \langle m_i \rangle$ is cyclic of order q and let

$$H = H/M_1 \times \ldots \times M_{k-1}$$

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then H is cyclic of order qr. Next we take W an irreducible and faithful IF_pH-module viewed as an H-module with $M_1 \times M_2 \times \ldots \times M_{k-1}$ in the kernel. Let

$$L = W^{M\langle x\rangle} = W \otimes 1 \oplus W \otimes x \otimes \ldots \otimes W \emptyset x^{r-1}$$

and form G = LM⟨x⟩ the semidirect product of L and M⟨x⟩. Then $C_w(x^r) = 1$ since it is an H-submodule not equal to W because the action of H is faithful. Also

$$(\omega \otimes x^i)x^r = \omega x^r \otimes r^i \qquad \forall \omega \in W \quad \text{and} \quad 0 \leq i \leq r-1$$

hence $W \otimes x^i \simeq W$ as x^r-module. Thus $C_L(x^r) = 1$. This implies that $C_L(x) = 1$.

Clearly, G is an A-group. To see why it is a CS_n group we first observe that

$$G = L \rtimes H$$

where H = M⟨x⟩ is a p′-metabelian group. By [(3.5) Makarfi, 1991], $H \in CS_n$. Next, any q element in H is conjugate to some $m \in M$ and by co-prime action we have

$$[L, \langle m\rangle^{..H}] \cap C_L(m) = 1$$

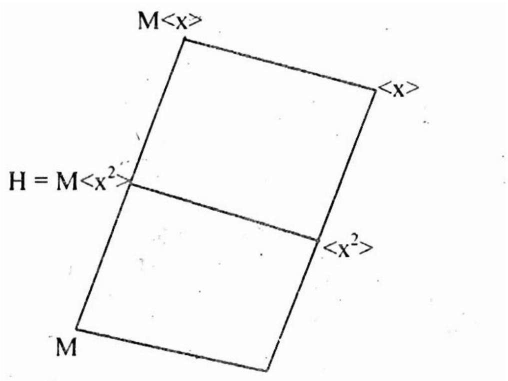
where $\langle m\rangle^{..H}$ denotes the smallest subnormal subgroup of H containing m. Note that $\langle m\rangle^{..H} = \langle m\rangle$ since M is abelian and normal in H. Also, any r-element of H is conjugate to an element in ⟨x⟩. But for $1 \neq y \in \langle x\rangle$, $y = x^n$ for some $n \in \{1,2, \ldots, r^2 = 1)$. If n = r we have seen that $C_L(x^r) = 1$. If $n \neq r$ then $y^r = x^{nr}$ and

$$C_L(y) \leq C_L(y^r) = C_L(x^{nr}) = 1.$$

Hence by theorem A of (Makarfi, 1991), G is a CS_n-group. The following special case gives an idea of what G looks like.

Let r = 2, q = 3 and k = 1 then M = ⟨m⟩ and ⟨x⟩ are cyclic of orders 3 and 4 respectively. Also,

$$M\langle x\rangle = \langle m,x \mid m^3 = x^4 = 1,\ m^x = m^{-1}\rangle.$$



$\overline{H} = H = \langle m\rangle \times \langle x^2\rangle$ has order 6. Let p = 7 and take $\dim IF_7W = 1$ so W is cyclic of order 7. Then

$$L = W \oplus 1 \oplus W \oplus x \text{ and } |L| = 7^2$$

such that $l^{x^2} = l^{-1} \forall l \in L$. Thus G = LM⟨x⟩ is a non-metabelian A-group in CS_n. In fact G is an A-group of nilpotent length three.

The above example shows that in theorem A below, if the generating elements are not of prime order, the result is not necessarily true.

The next result will be needed in the proof of theorem B.

2.3 Definition: A group is said to be monolithic if it has a unique minimal normal subgroup. The unique minimal normal subgroup is called the monolith of the group.

Lemma 2.3 Let G be a monolithic A-group in CS_n with monolith W which is a p-group. Then G has a p-subgroup P and a p'-subgroup H such that

$$G = P \rtimes H$$

and the Fitting subgroup F(H) of H contains all the elements of H of prime order.

Proof By 4(iv) (Makarfi, 1991), G splits over its Fitting subgroup which was shown to be a homocyclic p-group for some prime p. In fact G was shown to be a split extension of a p-group by a p'-subgroup. So we may assume that

$$G = P \rtimes H$$

where P is a p-group and H a p-group. It then remains to show that the Fitting subgroup of H contains all the elements of H of prime order. Now let

$$G_1 = \langle x \in G | 0(x) \text{ is prime} \rangle$$

then G_1 is metabelian by theorem A. Hence $G_1 \leq F_2(G)$. But

$$F_2(G)/F(G) = F(G), \text{ where } G = G/F(G)$$

$$F_2(G)/P = F(G/P) = F(H)$$

since F(G) = P by section 4 of (Makarfi, 1997). Thus $F_2(G) = PH_1$, where $H_1 = F(H)$ and the result follows.

Lemma 2.4 Let G be a monolithic group with monolith M and let k be a field of characteristic q and suppose that $q \dagger |M|$, then there exists an irreducible kG-module faithful for G.

Proof: We first denote the group algebra kG by W, viewed as a right module. Then W is a kG module faithful for G, since

$$1 \neq g \in G \Rightarrow l\omega.g = g$$

Hence $[W, M] \neq 0$ because if

$$1 \neq m \leq M, \exists\ \omega \in W \ni \omega m \neq \omega \text{ i.e. } \omega(m-1) \neq 0$$

since the action of G is faithful. Next let

$$0 = W_0 < W_1 < \dots < W_r = W$$

be a composition series for W as kG-module. If

$$[W_{i+1}/W_i, M] = 0 \ \forall i \in \{0, 1, \dots, r-1\}$$

then

$$[W_{i+1}, M] < W_i \ \forall i \in \{0, 1, \dots, r-1\}$$

$$\therefore [W, M, M, \dots, M] = [W, rM] = 0$$

Hence, as $q \dagger |M|$ we get

$[W, M] = 0$ a contradiction

so

$[W_{i+1}/W_i, M] \neq 0$ .... (*)

for some $i \in \{0, \ldots, r-1\}$

Let $X = W_{i+1}/W_i$ then X is an irreducible kG-module. Suppose that

$$K = \{g \in G \mid xg = x, \ \forall\, x \in X\}$$

then $K \triangleleft G$ since $K = C_G(X)$. Also $M \not\leq K$ by (*) so $K = 1$ since M is the monolith of G. Thus X is faithful.

Theorem 2.5 Suppose that $G = V_3G_2$ is a monolithic group with monolith V_3 which is an elementary abelian q-group, where $G_2 = V_2G_1$ is also monolithic with monolith V_2, an elementary abelian p-group and $G_1 = \langle x \rangle$ is cyclic of order r, such that q, p, r are distinct primes. Then $C_{V_3}(x) \neq 0$ i.e. x has a non-trivial centralizer in V_3.

Proof: We first note that Lemma (2.4) assures the existence of such a group.

Now $C_{V_3}(x) \neq 0$ if and only if x has an eigenvalue 1 on V_3. Let $k \geq \mathbb{F}_q$, be a splitting field for $G = V_3G_2$ and consider

$$\overline{V} = V_3 \otimes_{\mathbb{F}_q} k$$

Since the characteristic polynomial of x on V_3 is the same as that of x on $\overline{V}$, it is enough to show that x has eigenvalue 1 on $\overline{V}$.

But $\overline{V}$ is completely reducible as a kG_2-module, since $(|G_2|, p) = 1$ so we may suppose that

$$\overline{V} = \overline{V}_1 \oplus \ldots \oplus \overline{V}_t$$

where the $\overline{V}_i$, $1 \leq i \leq t$ are the irreducible submodules each faithful for G_2. To see that G_2 is faithful on the $\overline{V}$'s, for each $g \in G$ look at the action of the Galois group $k/\mathbb{F}_q$ on X(g) the matrix of g. If we assume that g operates trivially on any $\overline{V}_i$ then it operates trivially on V_3 since the Galois group operates transitively on $\{\overline{V}_1, \ldots, \overline{V}_t\}$. But then this implies that $g = 1$ since G_2 is faithful on V_3.

We next use Clifford's theorem, see for instance, [(3.4.1) of Gorenstein, 1980]. Let $U = \overline{V}_i$ and suppose that S is the inertia group of U_1 in G_2 where U_1 is an irreducible kV_2-submodule of U. Since S is a subgroup of G_2 containing V_2, $S = V_2$ or $S = G_2$. If $S = G_2$ then all the irreducible kV_2-submodules of U are isomorphic and so

$$U \simeq U_1 \oplus \ldots \oplus U_t$$

Now U_1 is 1-dimensional since V_2 is abelian. So $U_1 = ku_1$ for some $0 \neq u_1 \in U_1$. If

$y \in V_2$ then $u_1 y = \lambda_y u_1$ where $\lambda_y \in k^*$

i.e. the matrix of the action of y on U is of the form

$$\begin{pmatrix} \lambda_y & & 0 \\ & \ddots & \\ 0 & & \lambda_y \end{pmatrix}$$

So it commutes with that of every elements of G_2. As U is a faithful G_2-module this means that y is in the centre of G_2 i.e. $V_2 \leq Z(G_2)$. But this contradicts the fact that G_1 acts nontrivially on V_2. So we must have $S = V_2$ and this implies that U is isomorphic to the G_2-module induced from U_1 i.e.

$$U = U_1^{G_2} \text{ hence } U = \bigoplus_{i=0}^{r-1} U_1 \oplus x'$$

and if $0 \neq u_1 \in U_1$ *then*

$$0 \neq \sum_{i=0}^{r-1} u_1 \otimes x^i = u \in U$$

and u is fixed by x. Hence $Cv_3(x) \neq 0$.

3. MAIN RESULTS

In this section we prove the 3 main results in this paper. We start with the first result.

3.1. Theorem A: Let G be an A-group in CS_n such that G is generated by elements of prime order, then G is metabelian.

Before we go to the proof we bring two immediate consequences of this theorem. Note that if G is an A-group whose Sylow subgroups are elementary abelian then G is generated by elements of prime order, because of this we have the following.

3.2 Corollary. If G is an A-group in CS_n with elementary abelian Sylow subgroups then G is metabelian.

The above Corollary leads us to the following lemma which concerns CS-groups (Hartley, et al. 1982).

3.3 Lemma 3.3. Suppose that G is an A-group in CS_n with elementary abelian Sylow subgroups, then G is a CS-group.

Proof: This is a direct consequence of the above corollary and the fact that metabelian groups are CS-groups, see for instance (Poland, 1981).

We now go to the proof of the theorem. We know that the class of A-groups is S- and Q-closed, see (Taunt, 1949). Also by [(2.10) (Makarfi, 1991)], a minimum counter example can be assumed to be monolithic. Now the monolith, say W, is elementary abelian p-group for some prime p. The Fitting subgroup F = F(G) is also an abelian p-group, in fact F is homocyclic, see [Section 4 of Makarfi, 1997]. The next thing is to note that we can also assume that W = F otherwise G/W is metabelian by the minimality of our counter example. So the second Fitting subgroup, $F_2(G/W) = G/W$. Now by [(5.13) of Gorenstein, 1980].

$$W \leq F^P = \phi(F)$$

where $\phi(F)$ is the Frattini subgroup of F. So we have

$$F(G/W) = F(G)/W$$

by [(6.1.3) of Bechtell, 1971], thus

$$F_2(G/W)/F(G/W) = G/W/F(G)/W \simeq G/F(G)$$

is abelian and hence G is metabelian.

So we can consider W = F as a faithful irreducible G = G/W-module. Now suppose that

$$F_2(G)/F = F(G/F)$$

and note that $F_2(G) < G$ since G is not metabelian. Therefore

$\exists\ 1 \neq x \in G \backslash F_2(G)$

and since G is generated by elements of prime order we can choose x to be of prime order, say r. Let $F_2(G) = F_2$ and $\overline{F}_2 = F_2/F$ then G/F_2 acts faithfully on $\overline{F}_2$ and the Sylow subgroups of F_2 are normal in G/F. So for some prime $q \neq r$ there exists a q-subgroup $\overline{N}$ of $\overline{F}_2$ which is x-invariant.

Let $\overline{N} = M/F$ for some $M \leq F_2$. Then M is x-invariant and x leaves invariant some Sylow q-subgroup of M. Thus we can find an x-invariant q-subgroup N of M minimal subject to $[N, x] \neq 1$. By the minimality of N and co-prime action we must have $[N, x] = N$.

Since G is faithful on W it is clear that $[W,N] \neq 1$ and $[W,x] \neq 1$. We further note that N is elementary abelian. Now if we let

$$V_3 = [W, N],\ V_2 = N \text{ and } G_1 = \langle x \rangle$$

then by (2.5), $C_{V_3}(r) \neq 1$. Let $G_3 = V_3V_2G_1 = V_3N\langle x \rangle$ then

$$\langle x \rangle^{G_3} = G_3.$$

Thus G_3 is not a CS_n-group by [(3.2) of Makarfi, 1991] since $\langle x \rangle^{G}$ contains V_3 which contains a nontrivial centraliser of x. But then

$$G_3 \leq G \in CS_n$$

and we get a contradiction since CS_n is S-closed.

3.4 Theorem B: Suppose that G = W H is an A-group which is a semidirect product of an elementary abelian p-group W with a metabelian p′-group H. Suppose also that the action of H on W is faithful and irreducible and H′ is elementary abelian. Then G is a CS_n group if and only if every element of H of prime order is in $H_1 = F(H)$ the Fitting subgroup of H.

Proof: We first assume that $G \in CS_n$ and show that H_1 contains all the elements of H of prime order. But the hypothesis of the theorem implies that Lemma (2.3) is applicable and we see that H_1 contains all the elements of H of prime order. Conversely, we assume that every element of H of prime order is in H_1 and show that G is a CS_n group. Let $1 \neq h \in H$ be of order q^α for some prime q and integer $\alpha \geq 1$. By theorem A of (Makarfi, 1991) it is enough to show that

$$[W, \langle h \rangle^{H}] \cap C_W(h) = 1$$

since we know that H being metabelian is in CS_n. We can assume that $\alpha > 1$ for otherwise $h \in H_1$ and $\langle h \rangle^{H} = \langle h \rangle$ and the result follows by co-prime action.

Now let $h_1 = h^{q^{\alpha-1}}$, by hypothesis $h_1 \in H$. Since H splits over H′ then H = H′L for some complement L of H′ in H. Let H'_q and L_q denote the Sylow q-subgroups of H′ and L respectively. Then h is conjugate to xy for some $x \in H'_q$ and $y \in L_q$. Therefore

$$h_1 = h^{q^{\alpha-1}} = x^{q^{\alpha-1}}.y^{q^{\alpha-1}} = y^{q^{\alpha-1}}$$

since H′ is elementary abelian. Thus $h_1 \in L$ and hence $h_1 \in H_1 \cap L$. But both H_1 and L are abelian and $H' \leq H_1$, thus we have $h_1 \in Z(H)$ the center of H. This implies that $C_w(h_1)$ is an H-module, since W is the direct sum of the centralisers of the elements of H. But the action of H is faithful and irreducible. So $C_w(h_1) = 1$. But

$$C_w(h) \leq C_w(h_1) = 1$$

and the result follows.

3.5 Theorem C: Suppose that G = W ⋊ H is an A-group where W is a minimal normal p-subgroup of G for some prime p and H is a metabelian p′-subgroup acting faithfully on W. Then G is a CS_n group if and only if every element of H\F(H) acts fixed point freely on W, where F(H) is the Fitting subgroup of H.

Proof: By theorem A of (Makarfi, 1991), since $H \in CS_n$, to show that $G \in CS_n$ is enough to show that

$$[W, K] \cap C_w(h) = 1$$

where h is any q-element of H and $K = \langle h\rangle^{..H}$. Now if $h \in H\backslash F(H)$ then by hypothesis h acts fixed point freely on W. This means that h has no nontrivial centralizer in W i.e. $C_w(h) = 1$. So the result holds in this case. If on the other hand $h \in F(H)$ then $K = \langle h\rangle$ and by co-prime action the result again follows.

Conversely, suppose that $G \in CS_n$ then the problem is to show that every element $h \in H\backslash F(H)$ acts fixed point freely on W.

Note that it is enough to assume that h has prime power order. Since if the order of h is m then

$$C_w(h) \le C_w(h^t) \text{ for all } 1 \le t \le m-1$$

So to show that $C_w(h) = 1$ it is enough to show that $C_w(h^t) = 1$ for any $1 \le t \le m-1$. But then we can always choose t such that h^t has prime power order.

Also, for some complement L of H^t in H

$$H = H'L \text{ by (2.1)d of (Makarfi, 1997),}$$

where H' is the derived subgroup of H. Now let h be a q-element and $R = O_{q'}(H')$, the idea is to show that

$$K = \langle h\rangle^{..H} = [R,h]\langle h\rangle.$$

Let Q be a Sylow q-subgroup of H containing h then $H'Q = RQ$ is normal in H, since

$$H'Q/H' \triangleleft H/H'$$

Also, for any $y \in H$

$$[H,h]^y = [R,h[h,y]] \quad \text{since } R \triangleleft H$$

$$= [R,h] \quad \text{since } H' \text{ is abelian}$$

so that $[R,h] \triangleleft H$ and we have $[R,h]\langle h\rangle \triangleleft RQ \triangleleft H$

thus

$$K \subseteq [R,h]\langle h\rangle.$$

We also have

$$[F,h] \le [R,K] = [R,K,K] \le K^{H,n} = K$$

where n = s(H:K) is the defect of K in H. Next we note that

$$[H,h] \triangleleft H \Rightarrow [W,[R,h]] \triangleleft G$$

and since W is minimal normal in G, we must have

$$[W, [R,h]] = W \text{ or } 1.$$

But

$$[W, [R,h]] = 1 \Leftrightarrow [R,h] = 1$$

Hence H acts faithfully on W. Also $[R,h] = 1 \Leftrightarrow h \in F(H)$.

To see this we note that $R \leq F(H)$ which is abelian. So if $h \in F(H)$ then $[R,h] = 1$. On the other hand if $[R,h] = 1$ then $R\langle h\rangle$ is abelian and hence nilpotent, it is also subnormal in H, hence $R\langle h\rangle \subseteq F(H)$. Thus if $h \notin F(H)$ i.e. $h \in H \backslash F(H)$ then

$$[W, [R,h]] = W$$

$$\therefore [W, K] = W.$$

But then

$$[W,K] \cap C_w(h) = 1$$

so we must have $C_w(h) = 1$ and hence h acts fixed point freely on W. This completes the proof.

4. CONCLUDING REMARKS

We have seen that A-group with elementary abelian Sylow subgroup in CS_n are necessarily CS-groups since they are metabelian as (3.3) shows. This raises the question whether all A-groups in CS_n are also CS groups. Unfortunately, the answer to this question is negative. This is because Lemma 6 of Hartley, et al, 1982 implies that for the group in theorem C to be a CS-group every non-trivial element of H must necessarily act fixed point freely on W.

As a matter of fact, if one replaces L with some irreducible $M\langle x\rangle$-submodule K in Example (2.2) above one gets an example of an A-group in CS_n which is not a CS-group. Since the action of H on K is not fixed point free, W which is contained in K is fixed by many elements of H.

We have generally tried to get hold of all those A-groups in CS_n. So far the results we have in theorems B and C deal with only A-groups of nilpotent length three, i.e. an extension of an abelian group by a metabelian group. With these results we should next try to see if we can get a reasonable description of these nilpotent length three A-groups in CS_n.

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