GLOBAL JOURNAL OF MATHEMATICAL SCIENCES VOL. 5, NO. 1, 2006: 35 - 44
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MONOLITHIC A-GROUPS IN CS_n

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(Received 18 November, 2005; Revision Accepted 3 February, 2005)

ABSTRACT

The paper discusses monolithic A-groups in CS_n and gives the main results in Theorems A and B. Theorem A puts forward the necessary and sufficient conditions for a group that is a split extension of an abelian p-group V, for some prime p, by a group H to be in CS_n. V is considered as an irreducible kH module over $k = IF_p$ which is a splitting field for H. Thoerem B considered a monolithic A-group with monolith W, which implies that W is elementary abelian p-subgroup for some prime p and G is a split extension of a homocyclic p-subgroup P by a p'-subgroup H. It states that G is a CS_n group if and only if the subgroup G_1, a split extension of W by H, is a CS_n group. It further adds that if IF_p is a splitting field for H then the condition for G_1 to be in CS_n is given by Theorem A.

KEYWORDS: Monolithic A-group in CS_n, p-groups.

1. INTRODUCTION

In our discussion on CS_n A-group of nilpotent length three (Makarfi, 1997b) the issue of monolithic groups in CS_n came up. We also know that any class of finite groups that is S-, Q- and D- closed, Lemma (2.11) (Makarfi, 1991) contains exactly those groups that are subdirect product of the monolithic ones. It is therefore very clear that any serious investigation on CS_n groups must be based on good understanding of the monolithic ones. This underlines the motivation for the present discussions.

We bring our main results on monolithic A-group in CS_n in Theorems A and B on section 3.

2. PRELIMINARY RESULTS

In this section we look at those results that will help us to get to the main results that we shall bring in Section 3. We start with the following theorem.

2.1. Theorem (Theorem A of Makarfi (1991).

Suppose that P is a p-group and H is a group acting on P. Let $G = P \rtimes H$ be the semi-direct product of P by H then G is a CS_n-group if and only if

(a) H is a CS_n-group and

(b) $[P, \langle y\rangle^{\cdot\cdot H}] \cap C_P(y) = 1$

for every p' element y of H of prime order, where $C_p(y)$ is the centralizer of y in P and $\langle y\rangle^{\cdot\cdot H}$ is the subnormal closure of y in H.

The following is a well known result about subnormal subgroups which can be found in say chapter 13 of (Robinson, 1982).

2.2. Lemma (2.2)

Let $\{H_\lambda'/\lambda \in \Lambda\}$ be a family of subnormal subgroups of a group G such that for some integer n, $S(G: H_\lambda) \leq n$ for all λ. Then the intersection I of the H'_λ is subnormal in G and $S(G: I) \leq n$. In particular the intersection of any finite number of subnormal subgroups is again subnormal.

A full proof for the following result is given here

2.3 Lemma (2.3)

Let G be a group, H an abelian normal subgroup of G, V a kG-module, U a one (I)-dimensional kH-submodule of V with kernel A and y an element of G. Then

$$U \simeq Uy$$

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as kH-modules if and only if

$$[H,y] \leq A$$

Proof

Recall that $\Psi: U_1 \rightarrow U_2$ is a Kh-module isomorphism if and only if

1. Ψ is an isomorphism of vector spaces, and
2. $(u_1\Psi)h = (u_1h)\Psi, \ \forall h \in H$ and $u_1 \in U_1$

Now suppose that $[H,y] \leq A$. It is clear that $\Psi(u) = uy$ is a vector space isomorphism between U and Uy. So we onl need to show that

$$\Psi(uh) = \Psi(u)h \quad \forall h \in H$$

But

$$\Psi(uh) = uhy = u[h^{-1}, y^{-1}]\, yh = uyh \text{ since } [h^{-1}, y^{-1}] \in A$$

$$= \Psi(u)h.$$

Conversely, let

$\Psi: U \rightarrow Uu$ be an H-isomorphism.

We first show that

$$\theta : u \rightarrow uy$$

is also an H-isomorphism. Now for a fixed nontrivial element $u \in U$ there exists a non trivial element $\lambda \in k$ such that

$$\lambda\psi(u) = uy$$

This is because dimU = I = dim Uy. Since the map

$$x \rightarrow \lambda(\psi)(x)),\ x \in U$$

is also an H-isomorphism, we may assume that

$$\psi(u) = uy$$

then

$$\psi(\alpha u) = \alpha\psi(u) = \alpha uy = (\alpha u)y \quad \forall \alpha \in k$$

so

$$\psi(u) = uy \quad \forall u \in U$$

Therefore

$$u \rightarrow uy$$

is an H-isomorphism. Thus

$$\theta(uh) = \theta(u)h$$

Therefore

$$uhy = uyh \quad \forall h \in H$$

$$u[h^{-1}, y^{-1}] = u \ \forall h \in H$$

$$\Rightarrow \quad [H,y] \leq A$$

and this completes the proof.

We refer to the next result as co-prime action. Its proof can be found say in (5.3.6) of Gorenstein (1980).

2.4 **Lemma (2.4)**

Let A be a p'-group of automorphisms of a p-group P, then

$$[P, A, A] = [P,A]$$

where p is a prime.

2.5 **The next theorem reduces the proof of Theorem B considerably**

Theorem 2.5

Suppose that G is a monolithic A-group then we have the following

a) The monolith W is an elementary abelian p-group for some prime p.

(b) The Fitting subgroup P is a homocyclic p-group.

$C_G(W) = P$.

(c) $G = P \rtimes H$ for some p'-subgroup H of G.

Proof Refer to section 4 of Makarfi (1997a).

3. **THE MAIN RESULTS**

3.1 Our first main result gives us some kind of hold on those monolithic A-groups that are in CS_n.

Theorem A

Let H be a group and V a p-group which is an irreducible kH-module, where $k = IF_p$ is a splitting field for H and p is a prime. Suppose that

$$G = V \rtimes H$$

is an A-group, then G is in CS_n if and only if the following two conditions are satisfied

H is in CS_n and there exists

$$L \lhd K \leq H \text{ with } K/L \text{ cyclic}$$

and an irreducible and faithful k[K/L]-module U such that

$$V = U^H$$

L and K can be chosen such that L is subnormal in H and for all p'-elements, x in H, of prime power order

$$\langle x \rangle \cap K = \langle x \rangle \cap L \Leftrightarrow x \in L$$

Proof. Let G be in CS_n then we have to show (i) and (ii). As for (i), let F = F(H) be the Fitting subgroup of H. We first note that we can assume V to be faithful for H. Since if A is the kernel of the action of H on V and A is non trivial then |H/A| < |H| and G/A is in CS_n by the Q-closure property. Also V A/A is an irreducible k[H/A]-module, so that we can, by induction assume that

$$\exists\, L/A \lhd K/A \leq H/A$$

satisfying (i). But then $L \lhd K \leq H$ also satisfies (i)

So we can assume that V is faithful for H. Now the restriction $V|_F$ of V to F decomposes into irreducible kF-modules which are one (I)-dimensional, since F is abelian and k is a splitting field. We pick any of these one (I)-dimensional modules, say U_1. If the restriction is homogeneous then

$$V = U_1 \oplus \ldots U_1$$

and we see that each non trivial element of F acts as a scalar matrix on V. This matrix will commute with that of every other element of H, since H is faithful on V. But then F will be central in H and so $V = U_1$. Hence H = F and we can let K = F and L = 1.

This means we can assume that the decomposition of V into irreducible F-modules to be non-homogeneous. We next let T be the stabilizer of U_1 in H. Now T < H and if W is the Wedderburn component of V_F containing U_1, then

$$W \rtimes T < G$$

is in CS_n. So by induction on |G| we can assume that

$$\exists\, L \lhd K \leq T \text{ with } K/L \text{ cyclic}$$

and an irreducible k[K/L]-module U such that

$$W = U^T$$

where W is the Wedderburn component containing U_1 in the decomposition of $V|_F$. We also have

$$V \simeq W^H$$

We are here using Clifford's theorem (Clifford, 1937). Now because inducing is transitive we have

$$V = U^H$$

This completes the proof of (i). To show that (ii) also holds, we assume (i) and start by showing that L is subnormal in H. Note that since G is in CS_n then for any p'-element x in H of prime power order, we have

$$C_v(x) = C_v(\langle x\rangle^H)$$

by (2.4) and (2.1). Now because L centralizes U then for each x in L we have

$$u \otimes 1 \in C_v(x);\ \forall\, 0 \neq u \in U$$

Therefore

$$u \otimes 1 \in C_v(\langle x\rangle^H)$$

This means that every element of $\langle x\rangle^H$ centralizes $u \otimes 1$. Now for each $y \in L$ we have

$$(u \otimes 1)y = u \otimes y.1 = u \otimes 1$$

If on the other hand $y \in H$ and

$$(u \otimes 1)y = u \otimes 1$$

let $y = k_1 t$ for some $k_1 \in K$ and t an element of a transversal to K in H, then

$$U \otimes 1 = (u \otimes 1)\, y = (u \otimes 1)\, k_1 t = uk_1 \otimes t$$

and we see that $k_1 \in L$ and t = 1. Thus

$$(u \otimes 1)y = u \oplus 1 \Leftrightarrow y \in L$$

Hence

$$\langle x\rangle^{\cdot\cdot H} \leq L$$

As x runs through L we see that L is generated by these subnormal subgroups of H. Thus L is also subnormal in H, since it is generated by a finite number of subnormal subgroups by (2.2).

It now remains to show that if x is a p'-element of H of prime power order then

$$\langle x\rangle \cap K = \langle x\rangle \cap L \Leftrightarrow x \in L.$$

So we suppose that x is an element of H of order q^α for some prime $q \neq p$ and some integer $\alpha \geq 1$. We also suppose that

$$\langle x^n\rangle = \langle x\rangle \cap K = \langle x\rangle \cap L;\ 1 \leq n \leq q^\alpha$$

The problem now is to show that x is in L. First of all x centralizes

$$u = u \otimes 1 + u \otimes x + \ldots + u \otimes x^{n-1}$$

where $0 \neq u \in U$. Now since

$$C_v(x) \leq C_v(\langle x\rangle^{\cdot\cdot H})$$

it follows that every element y in $\langle x\rangle^{\cdot\cdot H}$ also centralizes v. Because 1, x, …, x^{n-1} is part of a transversal to K in H, we have

$$vy = v$$

$$\Leftrightarrow u \otimes 1 + u \otimes x + \ldots + u \otimes x^{n-1} = (u \otimes 1)y + u \otimes xy + \ldots + u \otimes x^{n-1}y$$

$$\Leftrightarrow (u \oplus 1)yx^{-i} = u \otimes 1 \text{ --- } (^{**}) \text{ for some } i \in \{0,1, \ldots, n-1\}$$

i.e. y = $\ell_i x^i$ for some $\ell_i \in L_n\langle x\rangle^{\cdot\cdot H}$

Since both y and x^i are elements of $\langle x\rangle^{\cdot\cdot H}$. Thus

$$M = \langle x\rangle^{\cdot\cdot H} = (L_1, r)$$

where $L_1 = L \cap \langle x\rangle^{\cdot\cdot H}$ This is because (**) is true for all y in $\langle x\rangle^{\cdot\cdot H}$

Next we let

$$\overline{H} = H / O_{q'}(H) \text{ and } \overline{F} = F(\overline{H})$$

then $\overline{F}$, being the Fitting subgroup of $\overline{H}$, has no non trivial q'-elements since $O_{q'}(\overline{H})$ is trivial. Also it has to contain all the q-elements, since it is its own centralizer in $\overline{H}$ and $\overline{H}$ is an A-group. So the inverse image of $\overline{F}$ in H is $O_{q'}(H)Q$ is some Sylow q-subgroup of H. Thus

$$O_{q'}(H)\, Q \lhd H.$$

Without loss of generality we can assume that x is in Q. Now it is clear that

$$M \leq [O_{q'}(H), x]\langle x\rangle$$

because

$$[O_{q'}(H), x]\, \langle x\rangle \lhd O_{q'}(H)Q.$$

We use induction on $|O_{q'}(H)|$ to show that

$$[O_{q'}(H), x] \leq M.$$

By co-prime action x leaves invariant some Sylow r-subgroup B of $Q_{q'}(H)$, let R be some r′-subgroup of $O_q(H)$ such that $O_{q'}(H) = \langle R, B\rangle$. Now again by co-prime action

$$[B,x] = [B, x, \dots, x] \leq [H, x, \dots, x]$$
$$\leq [H, M, \dots, M]$$
$$< M$$

where x and M appear s(H:M) times.

By induction we assume that $[R,x] \leq M$, so that

$$[O_{q'}(H), x] = \langle [R, x], [B. x]\rangle \leq M.$$

Therefore

$$M = [O_{q/'}(H), x]\langle x\rangle = \langle L_1, x\rangle$$

Thus

$$[O_{q'}(H), x] \leq \langle L, x\rangle.$$

Also for all y in $[O_{q'}(H), x]$ there exists x^i by (**), such that

$$yx^i \in L$$

i.e. $$yx^i \in (O_{q'}(H)Q) \cap L$$

But $Q \cap L$ is a Sylow q-subgroup of L since L is subnormal in H. So

$$O_{q/}(H) \cap L = O_{q'q}(L) = O_{q'}L.(Q \cap L)$$
$$= (O_{q'}(H) \cap L)(Q \cap L)$$

This gives us

$$yx^i \in (O_{q'}(H) \cap L.(Q \cap L).$$

Therefore

$$yx^i = uv$$

for some $u \in O_{q'}(H) \cap L$ and $v \in Q \cap L$. But

$$u^{-1}y = vx^{-1} \in Q_{q/}(H) \cap Q = 1$$

thus $y \in L$ and we get

$$[O_{q/}(H), x] \leq L$$

so that

$$[O_{q/}(F), x] \leq L \cap F = L_o. \qquad (a)$$

Now L_o is the kernel of F on U_1 and by (2.3) we have

$$T = C_H(F/L_o). \qquad (b)$$

The next thing to observe is that equations (a) and (b) imply that x is in T because

$$F = O_{q/}(F)(F \cap Q)$$

so that

$$[F,x] = [O_{q'}(F), x] \leq L_o.$$

We next note that

$$W \rtimes T < V \rtimes H = G$$

and G is in CS_n implies that

$$W \rtimes T \in CS_n$$

because CS_n is S-closed. Thus by induction on |G| we can conclude that x is in L_o and hence in L.

Conversely, let

$$G = V \rtimes H$$

We assume (i) and (ii) and show that G is a CS_n group. From (i) we know that H is a CS_n group. So by (2.1) it is enough to show that for any p′-element x in H of prime power order

$$[V, \langle x\rangle^{H}] \cap C_v(x) = 0 \qquad \text{(c)}$$

Now to show (c) it is enough, by co-prime action, to show that

$$Cv(x) \leq Cv(\langle x\rangle^{H})$$

for any such x. We may assume that Cv (x) ≠ 0 for otherwise (c) trivially holds. We know that

$$V|_{\langle x\rangle} = \bigoplus_{t} U \otimes t|_{\langle x\rangle \cap K^t}| \langle x\rangle$$

$$\bigoplus_{t} V_t, \text{ say}$$

and $C_v(x) = \oplus_t Cv_t(x)$

where t runs through T_x which is some transversal to (K, ⟨x⟩) double cosets.

Now $U \otimes t$ is a K^t module with kernel L^t. So if

$$\langle x^m\rangle = \langle x\rangle \cap K^t = \langle x\rangle \cap L^t$$

then x is fixed point free on V_t. To see this let

$$0 \neq v \in Cv_t(x)$$

and suppose that

$$v = u_1 \otimes t + u_2 \otimes tx + \ldots + u_m \otimes tx^{m-1}$$

where $u_i \in U$ for $1 \leq i \leq m$. Then x^m fixes v so that

$$\left(\sum_{i=1}^{m} u_i \otimes tx^{i-1}\right)x^m = \sum_{i=1}^{m} u_i \otimes tx^{i-1}.x^m$$

i.e.

$$\sum_{i=1}^{m} u_i \otimes tx^{i-1}x^m = \sum_{i=1}^{m} u_i tx^m t^{-1} \otimes tx^{i-1} = \sum_{i=1}^{m} u_i \oplus tx^{i-1}$$

Therefore

$$u_i tx^m t^{-1} = u_i \text{ for } 1 \leq i \leq m.$$

Therefore

$$u_i = 0 \text{ since } tx^m t^{-1} \in K\backslash L.$$

If on the other hand

$$\langle x\rangle \cap K^t = \langle x\rangle \cap L^t = \langle x^n\rangle$$

then $U \otimes t$ is a trivial $\langle x\rangle \cap L^t$-module and so x has a fixed point in V_t since

$$v = u \otimes t + (u \otimes t)\, x + \ldots + (u \otimes t)x^{n-1}$$

is fixed by x for any $0 \neq u \in U$.

It is then clear from the above discussion that for any x in H of prime power order

$$0 \neq Cv_t(x) \Leftrightarrow \langle x\rangle \cap K^t = \langle x\rangle \cap L^t$$

for some $t \in T_x$. Thus

$$0 \neq Cv_t(x) \Rightarrow \langle x\rangle \cap K^t = \langle x\rangle \cap L^t$$

$$\Rightarrow \langle x^{t-1}\rangle \cap K = \langle x^{t-1}\rangle \cap L$$

$$\Rightarrow x^{t-1} \in L \text{ by (ii)}$$

$$\Rightarrow x \in L^t$$

$$\Rightarrow \langle x\rangle^{..H} \in L^t,$$

$$\Rightarrow Cv_t(x) \leq Cv(\langle x\rangle^{..H}).$$

But

$$Cv(x) = \oplus Cv_t(x)$$

Hence

$$Cv(x) \leq Cv(\langle x\rangle^{..H}).$$

3.2 We now come to theorem B and the proof.

Theorem B

Let B be a monolithic A-group with monolith W, so that by Theorem (2.5), W is elementary abelian p-group for some prime p and

$$G = P \rtimes H$$

where P is a homocyclic p-group and H is a p'-subgroup. Then G is in CS_n if and only

$$G_1 = W \rtimes H$$

is in CS_n. If $\mathbb{F}_P$ is a splitting field for H then the condition for G_1 to be in CS_n is given by theorem A.

Proof. If G is in CS_n then G_1 is in CS_n by the S-closure property. Now W can be considered as an $\mathbb{F}_P H$-module and since W is the monolith we see that G_1 satisfies the hypotheses of theorem A and is applicable.

On the other hand if G_1 is in CS_n we have to show that G is in CS_n. But by (2.1) it is enough to show that for each element h in H of prime power order

$$[P, \langle h\rangle^{..H}] \cap C_P(h) = 1$$

Now, G_1 is in CS_n implies that for each such element we have

$$[W, \langle h\rangle^{H}] \cap C_W(h) = 1. \qquad (*)$$

But if

$$[P, \langle h\rangle^{H}] \cap C_P(h) \neq 1.$$

then

$$1 \neq M = W \cap C_P(h) \cap [P, \langle h\rangle^{H}]$$

since W = $\Omega_1(P)$. This means that

$$M \leq W \cap C_P(h).$$

Therefore

$$M \leq C_W(h). \qquad (**)$$

Also

$$M \leq [P, \langle h\rangle^{H}]$$

and by co-prime action we have

$$[P, \langle h\rangle^{H}] = [P, \langle h\rangle^{H}, \langle h\rangle^{H}]$$

Therefore

$$M = [M, \langle h\rangle^{H}]$$

Lastly, $M \leq W$ and so

$$M = [M, \langle h\rangle^{H}] \leq [W, \langle h\rangle^{H}]$$

Using (**) we get

$$1 \neq M \leq C_W\langle h\rangle \cap [W, \langle h\rangle^{H}]$$

contradicting (*).

4. CONCLUDING REMARKS

The monolithic groups play a very crucial role in respect of any class of groups that is S-, Q- and D- closed. Since CS_n satisfies these three properties, theorem A is very decisive for any discussion on A-groups in CS_n.

The theorem has given us a reasonable description of the characteristics of the monolithic groups in CS_n. We should for instance be able to tackle the question of the bounds on the nilpotent length of A-groups in CS_n. The question on bounds on nilpotent length has two aspects. We may want to know whether there exists an integer n such that any A-group of nilpotent length greater than n can not be in CS_n, in other words all A-groups in CS_n must have nilpotent length less or equal to n.

On the other hand we have seen in Makarfi (1997a, 2005) that CS_n A-groups whose Sylow subgroups are generated by elements of prime order are metabelian, i.e. they are of at most nilpotent length 2. This means that the internal structure of the group may also affect the nilpotent length.

ACKNOWLEDGEMENT

The author wishes to thank the International Centre for Theoretical Physics, Trieste for giving him the facilities to prepare this paper.

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