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ON DIFFERENTIAL OPERATORS ON $W^{1,2}$ SPACE AND FREDHOLM OPERATORS

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ABSTRACT
A selfadjoint differential operator defined over a closed and bounded interval on Sobolev space which is a dense linear subspace of a Hilbert space over the same interval is considered and shown to be a Fredholm operator with index zero.

KEYWORDS: Sobolev space, Hilbert space, dense subspace, Fredholm operator

1.0 INTRODUCTION

We shall consider the differential operator which is selfadjoint (Egwurube and Garba, 2001) defined on a Hilbert space as follows

$$A(\xi(t)) = J\frac{d\xi}{dt} + S(t)\xi(t),\ S^T = S \qquad (1.1)$$

where $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$, $D(A) = \{\xi \in W^{1,2}[0,1;R^{2n}], \xi(0) = \xi(1)\}$, $H = L^2[0,1;R^{2n}]$ and W=D(A).

Here D(A) is the domain of A.

Egwurube and Garba,(2002); in their paper generalized the above problem (1.1) by considering

$$F(\xi(t)) = \frac{d\xi}{dt} + A(t)\xi(t) \qquad (1.2)$$

where $A(t) \in L(W,H)$ is selfadjoint and continuously differentiable with W a dense linear subspace of H and showed that the operator (1.2) is Fredholm provided W and H are Hilbert spaces with $W \subset H$ a compact embedding and the limit operator $A^{\pm} = \lim_{t\to\pm} A(t)$ is bijective. Salamon, (1990); considered a partial differential operator problem $F: X(s,t) \to Y(s,t)$ defined by

$$F(\xi(s,t)) = \frac{\partial\xi}{\partial t} + J\frac{\partial\xi}{\partial s} + S(s,t)\xi(s,t), S^T = S \qquad (1.3)$$

where $X(s,t) = \{\xi(s,t) \in W^{1,2}([0,1]\times R; R^{2n}), \xi(0,t) = \xi(1,t), t \in R\}$ with J as defined in (1.1) and $Y(s,t) = L^2([0,1]\times R; R^{2n})$ showing that if $\lim_{t\to\pm}\sup|S(s,t) - S^{\pm}(s)| = 0$ then F is Fredholm.

Here $S^{\pm} = \lim_{t\to\pm} S(s,t)$.

In this paper we shall show that (1.3) is a special case of (1.2) and also that the operator (1.1) is Fredholm with index zero.

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2.0 Some Preliminaries and Fundamental Theorems.

Theorem 2.1

The space $C_0(\Omega)$ is dense in $L^1(\Omega)$, that is, for all $f \in L(\Omega)$ and for all $\varepsilon > 0$, there exists $f_1 \in C_0(\Omega)$ such that $\|f - f_1\|_{L^1} < \varepsilon$. In fact, this space is dense in $L^P(\Omega), 1 \le P < \infty$ (Adams,1978;Brezis,1983;Rudin,1987).

Proof

Given $\varepsilon > 0$ and $f \in L(\Omega)$, then there exists step functions $\varphi : \Omega \to \Re$ such that $\varphi \to f$ almost everywhere and $\int_\Omega |\varphi - f| dx \le \frac{\varepsilon}{2}$. Suppose also that there exists continuous functions $f_1 \in C_0(\Omega)$ such that $\int_\Omega |\varphi - f_1| dx \le \frac{\varepsilon}{2}$.

Then $\|f - f_1\|_{L^1(\Omega)} \le \|f - \varphi\|_{L^1(\Omega)} + \|\varphi - f_1\|_{L^1(\Omega)} \le \varepsilon$.

Definition 2.2

A function $x : [a,b] \to \Re$ is called absolutely continuous if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\sum_{j=1}^{n} |t_j - s_j| < \delta \Rightarrow \sum_{j=1}^{n} |x(t_j) - x(s_j)| \le \varepsilon$,

for all $a \le s_1 < t_1 \le s_2 < t_2 \le ... \le s_n < t_n \le b$. If $\xi : [a,b] \to \Re$ is integrable then $x(t) = \int_a^t \xi(s) ds$ is absolutely continuous.

Theorem 2.3 (Riesz-Nagy, 1952)

If $x : [a,b] \to \Re$ is absolutely continuous then $\dot{x}(t) = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}$ exists almost everywhere, moreover, $\dot{x} : [a,b] \to \Re$ is integrable and $x(t) = x(a) + \int_a^t \dot{x}(s) ds$.

Definitions and Notations 2.4

Let $W^{1,2}$ [a, b] = $\{x : [a,b] \to C \,|\, x \textit{ absolutely continuous (a.e) and } \dot{x} \in L^2\}$. Also defined as

$$W^{1,2}[a,b] = \{x : [a,b] \to C : \exists \xi \in L^2[a,b] \textit{ and } x_0 \in C \textit{ such that } x(t) = x_0 + \int_a^t \xi(s) ds\}.$$

This second definition follows from Theorem 2.5 below.

Theorem 2.5

Let $x, \xi \in L^2[a,b]$. The following are equivalent

(i) x is absolutely continuous and $x(t) = x(a) + \int_a^t \dot{x}(s)ds$

(ii) For every $\varphi \in C_0^1[a,b]$, $\int_a^b x(t)\dot{\varphi}(t)dt = -\int_a^b \xi(t)\varphi(t)dt$.

(iii) $\lim_{0<h\to 0} \int_a^{b-h} \left| \frac{x(t+h)-x(t)}{h} - \xi(t) \right|^2 dt = 0$.

Remark 2.6

If $x \in C^1, \xi = \dot{x}$, then (i) follows by partial integration.

Proof of Theorem 2.5

(i)$\Rightarrow$(iii)

Define $\xi(t) := 0$ for t > b, x(t): = x(b), t > b

$$\int_a^{b-h} \left| \frac{x(t+h)-x(t)}{h} - \xi(t) \right|^2 dt \le \int_a^b \left| \frac{x(t+h)-x(t)}{h} - \xi(t) \right|^2 dt = \int_a^b \left| \frac{1}{h} \int_0^h (\xi(t+s) - \xi(t))ds \right|^2 dt$$

$$= \frac{1}{h^2} \int_a^b \left| \int_0^h (\xi(t+s) - \xi(t))ds \right|^2 dt \le \frac{1}{h} \int_a^b \int_0^h |\xi(t+s) - \xi(t)|^2 dsdt = \frac{1}{h} \int_0^h \int_a^b |\xi(t+s) - \xi(t)|^2 dtds$$

$$\le \sup_{0<s<h} \int_a^b |\xi(t+s) - \xi(t)|^2 dt \to 0 \text{ as } h \to 0.$$

(iii)$\Rightarrow$(ii)

Let $\varphi \in C^1$ with support in [a, b]. This implies that

$$\lim_{h\to 0} \frac{\varphi(t) - \varphi(t-h)}{h} = 0 \Rightarrow \left| \int_a^b (x(t)\dot{\varphi}(t) + \xi(t)\varphi(t))dt \right|$$

$$= \lim_{h\to 0} \left| \int_a^b (x(t)\frac{\varphi(t) - \varphi(t-h)}{h} + \xi(t)\varphi(t))dt \right|$$

$$= \lim_{h\to 0} \left| \left(\int_a^b \frac{x(t)\varphi(t)}{h} dt - \int_{a+h}^b \frac{x(t)\varphi(t-h)}{h} dt + \int_a^b \xi(t))\varphi(t)dt \right) \right|$$

$$= \lim_{h\to 0} \left| \left(\int_a^{b-h} \frac{x(t)\varphi(t)}{h} dt - \int_a^{b-h} \frac{x(t+h)\varphi(t)}{h} dt + \int_a^{b-h} \xi(t))\varphi(t)dt \right) \right|$$

$$= \lim_{h\to 0} \left| -\int_a^{b-h} \left| \frac{x(t+h) - x(t)}{h} - \xi(t) \right| \varphi(t)dt \right|$$

$$\leq \lim_{h\to 0}\sqrt{\int_a^{b-h}\left|\frac{x(t+h)-x(t)}{h}-\xi(t)\right|^2 dt}\sqrt{\int_a^{b-h}|\varphi(t)|^2 dt}=0,\varphi\in C_0^1[\text{a, b}]$$

(ii)$\Rightarrow$(i)

Let X_0 = constant functions in L^2[a,b] such that $\{\psi\in C_0[a,b]:\int_a^b\psi(t)dt=0\}$ is dense in $X_0^{\perp}$.

This implies that $\{\psi\in C_0[a,b]:\int_a^b\psi(t)dt=0\}^{\perp}=X_0$. Let $\varphi(t):=\int_a^t\psi(s)ds$, then by (ii) above

$$0=\int_a^b(\bar{x}(t)\dot{\varphi}(t)+\bar{\xi}(t)\varphi(t))dt=\int_a^b\bar{x}(t)\psi(t)dt-\int_a^b\bar{\xi}(s)\int_s^b\psi(t)dtds$$

$$=\int_a^b(\bar{x}(t)\psi(t)-\int_a^b\int_a^t\bar{\xi}(s)ds\,\psi(t)dt=\int_a^b(\bar{x}(t)-\int_a^t\bar{\xi}(s)ds)\psi(t)dt$$

This implies that $x(t)-\int_a^t\xi(s)ds\perp\{\psi\in C_0[a,b]:\int_a^b\psi(t)dt=0\}$ which shows that

$x(t)-\int_a^t\xi(s)ds=constant$.

Theorem 2.7

(i) $W^{1,2}$[a,b] is a Hilbert space

(ii) C^1[a,b] is dense in $W^{1,2}$[a,b]

Proof

(i) Let $x_n\in W^{1,2}[a,b]$ be Cauchy with respect to the norm $\|\cdot\|_{W^{1,2}}$ then both x_n and $\dot{x}_n$ are Cauchy sequences in L^2[a,b], hence there exists $x,\xi\in L^2[a,b]$ such that $x_n\to x$ and $\dot{x}_n\to\xi$ in the L^2-norm. Therefore

$\int_a^b(\bar{x}(t)\dot{\varphi}(t)+\bar{\xi}(t)\varphi(t))dt=\lim_{n\to\infty}\int_a^b(\bar{x}_n(t)\dot{\varphi}(t)+\bar{\dot{x}}_n(t)\varphi(t))dt=0$ by Theorem 2.5.

This shows that $x\in W^{1,2}[a,b]$ *and* $\xi=\dot{x}$, therefore

$$\|x-x_n\|_{W^{1,2}}^2=\|x-x_n\|_{L^2}^2+\|\xi-\dot{x}_n\|_{L^2}^2\to 0$$

(ii) Let $x\in W^{1,2}[a,b],\varepsilon>0$, then there exists $\eta\in C[a,b]$, such that

$\|\eta-\dot{x}\|_{L^2}\leq\varepsilon$. Define $y(t)=x(a)+\int_a^t\eta(s)ds, y\in C^1[a,b]$. Consider

$$|y(t)-x(t)| = \left|\int_a^t (\eta(s)-\dot{x}(s))ds\right|$$

$$\leq \int_a^b |\eta(s)-\dot{x}(s)|ds \leq \sqrt{b-a}\|\eta-\dot{x}\|_{L^2} \leq \sqrt{b-a}\varepsilon$$

Therefore $\|y-x\|_{L^2} = \sqrt{\int_a^b |y-x|^2 dt} \leq (b-a)\varepsilon$, and

$$\|y-x\|_{W^{1,2}} \leq \sqrt{\|y-x\|_{L^2}^2 + \|\dot{y}-\dot{x}\|_{L^2}^2} \leq \varepsilon\sqrt{(b-a)^2+1}$$

Theorem 2.8

(i) There exists c > 0 for all x in $W^{1,2}$ [a,b] such that $\|x\|_{L^\infty} \leq c\|x\|_{W^{1,2}}$

(ii) Let $x_n \in W^{1,2}[a,b]$ be a sequence such that $\|x_n\|_{W^{1,2}} \leq c\ \forall\, n \in \mathbb{N}$ for some c > 0, then there exists a uniformly converging subsequence of x_n.

Proof

(i) $$|x(t)| = \left|x(s)+\int_s^t \dot{x}(\tau)d\tau\right| \leq |x(s)| + \int_a^b |\dot{x}(\tau)|d\tau$$

$$\leq |x(s)| + \sqrt{b-a}\|\dot{x}\|_{L^2}$$

Integrating both sides with respect to s over [a, b], we have

$$(b-a)|x(t)| \leq \int_a^b |x(s)|ds + (b-a)^{\frac{3}{2}}\|\dot{x}\|_{L^2}$$

$$\leq \sqrt{b-a}\left(\|x\|_{L^2} + (b-a)\|\dot{x}\|_{L^2}\right)$$

Therefore $|x(t)| \leq c\|x(t)\|_{W^{1,2}}$ *where* $c = (b-a)^{\frac{1}{2}}$

(ii) The proof makes use of the Arzela-Ascoli Theorem, that is, a subset $K \subset C[a,b]$ is compact if and only if it is closed, bounded and equicontinuous.

A family K of functions is said to equicontinuous on an interval I$\subseteq R$ if for all $\varepsilon > 0\ \exists\, \delta > 0$ such that $|x(t)-x(s)| < \varepsilon$ whenever $|t-s| < \delta, t, s \in I$ *and* $x \in K$.

Let $K = closure\{x_n : n \in N\}$ in C[a, b]. K is bounded by Theorem 2.8 (i) above and closed by definition. It remains to show that K is equicontinuous.

Suppose $x_n \in K$, then

$$|x_n(t)-x_n(s)| = \left|\int_s^t \dot{x}_n(\tau)d\tau\right| \le \int_s^t |\dot{x}_n(\tau)d\tau|$$

$$\le \sqrt{t-s}\sqrt{\int_s^t |\dot{x}_n|^2 d\tau} \le \sqrt{t-s}\|\dot{x}_n\|_{L^2} \le c\sqrt{t-s}$$

since $\|\dot{x}_n\|_{L^2[a,b]} \le \|x_n\|_{W^{1,2}[a,b]}$.

Definition 2.9 (Schecter ,1971)

(a) Let X and Y be Banach spaces. An operator $F \in L(X,Y)$ is called Fredholm if (i) dimker F is finite (ii) rangeF is closed in Y (iii) dimcokerF is finite.
(b) The index of a Fredholm operator F is defined as indexF=dimkerF – dimcokerF.

3.0 Main Results

Remarks 3.1
We shall now show that the operator F as defined by equation (1.3) is a special case of (1.2) by rewriting (1.3) as

$$F(\xi(s,t)) = \frac{\partial\xi}{\partial s} + A(t)\xi(s)$$

where $A(t): W^{1,2}_{per}([0,1], R^{2n}) \to L^2([0,1], R^{2n}]$ defined by

$$(A(t)\xi)(s) = J\frac{\partial\xi}{\partial s} + S(s,t)\xi(s)$$

Here $W^{1,2}_{per}([0,1], R^{2n}) = \{\xi \in W^{1,2}([0,1], R^{2n}); \xi(0) = \xi(1)\}$.

If we let H= $L^2[0,1]$ and W= $W^{1,2}_{per}[0,1]$ together with the fact that $W^{1,2}_{per}[0,1]$ is a continuous dense injection into $L^2[0,1]$ (Adams,1978), we obtain

$$\|\xi\|^2_{W^{1,2}(R,L^2[0,1])} = \int_{-\infty}^{\infty}(\|\xi\|^2_{L^2[0,1]} + \left\|\frac{\partial\xi}{\partial t}\right\|^2_{L^2[0,1]})dt$$

$$= \int_{-\infty}^{\infty}\int_0^1(|\xi(s,t)|^2 + \left|\frac{\partial\xi}{\partial t}(s,t)\right|^2)dsdt$$

Similarly

$$\|\xi\|^2_{L^2(R,W^{1,2}_{per}[0,1])} = \int_{-\infty}^{\infty}\int_0^1(|\xi(s,t)|^2 + \left|\frac{\partial\xi}{\partial t}(s,t)\right|^2)dsdt$$

Therefore

$$\|\xi\|^2_X = \int_{-\infty}^{\infty}\int_0^1(|\xi(s,t)|^2 + \left|\frac{\partial\xi}{\partial s}\right|^2 + \left|\frac{\partial\xi}{\partial t}\right|^2)dsdt = \|\xi\|^2_{W^{1,2}([0,1]\times R,R^{2n})}$$

where X = $W^{1,2}(R, L^2[0,1]) \cap L^2(R, W^{1,2}[0,1])$. Also if Y=$L^2(R,L^2[0,1])$ then

$$\|\xi\|_Y^2 = \int_{-\infty}^{\infty}\|\xi\|_{L^2[0,1]}^2 dt = \int_{-\infty}^{\infty}\int_0^1 |\xi(s,t)|^2 ds dt = \|\xi\|^2_{L^2([0,1]\times R, R^{2n})}$$

We shall now show that the operator defined by equation (1.1) is Fredholm which requires the following lemma.

Lemma 3.2 (Egwurube and Garba ,2001)

Let X,Y and Z be Banach spaces. Suppose $F \in L(X,Y)$ is a bounded linear operator and $K \varepsilon L(X,Z)$ a compact linear operator.If

$$\|x\|_X \leq c\,[\,\|Fx\|_Y + \|Kx\|_Z\,], \forall x \in X$$

where c is a constant then F has a closed range and finite dimensional kernel.

Proof: It suffices to show that the unit ball in kerF is compact to show that dimKerF is finite.Let $B = \{ x \in X : Fx = 0 , \|x\| \leq 1 \}$. Consider $x_n \in B$ then there exists a subsequence such that Kx_{nk} converges since K is compact. Therefore

$$\| x_{nk} - x_{nl} \|_X \leq c \| Kx_{nk} - Kx_{nl} \|_Z \to 0 \text{ as } k, l \to \infty$$

Thus x_{nk} is cauchy and because X is complete, $x_{nk} \to x \in X$. Therefore B is compact.

Let $y_n = Fx_n \in$ Range F such that $y_n \to y$,it remains to prove that $y \in$ Range F to show that Range F is closed. Suppose there exists a sequence $\xi_n \in$ Ker F such that $x_n + \xi_n$ is bounded. Hence there exists a subsequence $\ddot{y}_{nk} = x_{nk} + \xi_{nk}$ such that $K\ddot{y}_{nk} \to z$. Therefore $F\ddot{y}_{nk} \to y$ by our assumption. Hence $\ddot{y}_{nk}$ is cauchy and $\ddot{y}_{nk} \to x \in X$ and $y = \lim F\ddot{y}_{nk} = Fx$ which implies that $y \in$ Range F.

We shall now show that there exists a sequence $\xi_n \in$ Ker F such that $x_n + \xi_n$ is bounded . Suppose not, then $\inf \| x_n + \xi_n \| = c_n$ has unbounded sequence. Without loss of generality $c \to \infty$ .We choose ξ_n such that $c_n \leq \| x_n + \xi_n \| \leq 2c_n$ then $K(x_n + \xi_n)/c_n$ has a converging subsequence and

$$F(x_n + \xi_n)/c_n = F(x_n)/c_n \to 0 \text{ as } n \to \infty$$

Therefore $(x_n + \xi_n)/c_n$ is cauchy and converges to some $x \in X$ and

$$Fx = \lim_{n\to\infty} F(x_n + \xi_n)/c_n = 0$$

Hence for $\xi \in$ Ker F we see that

$$\|x + \xi\| = \lim_{n\to\infty}\left\|\frac{x_n + \xi_n}{c_n} + \xi\right\| \geq 1$$

contradicting the fact that $\xi \in$ Ker F.

Remarks 3.3

(i) The operator A as defined by equation (1.1) has closed range and finite dimensional kernel since

$$\begin{aligned}
\|\xi\|_W^2 &= \|\xi\|_H^2 + \|\dot{\xi}\|_H^2 = \|\xi\|_H^2 + \|J\dot{\xi}\|_H^2 \\
&= \|\xi\|_H^2 + \|J\dot{\xi} + S\xi - S\xi\|_H^2 \\
&\leq \|\xi\|_H^2 + (\|A\xi\|_H + \|S\xi\|_H)^2 \\
&\leq \|\xi\|_H^2 + 2(\|A\xi\|_H^2 + \|S\xi\|_H^2) \\
&\leq k(\|A\xi\|_H^2 + \|\xi\|_H^2) \text{ where } k = 2\left(1 + \sup_t \|S(t)\|_H^2\right)
\end{aligned}$$

$$\text{Therefore } \|\xi\|_W \leq c\left(\|A\xi\|_H + \|\xi\|_H\right), \; c = k^{\frac{1}{2}}$$

And by comparison with *lemma* 3.2: X = W , Y = H ,Z = H and F = A : X Y with K the injection of W into H, we see that range A is closed and dimKerA is finite.

(ii) Egwurube and Garba,(2002); showed that the operator A as defined by (1.1) is also selfadjoint, that is $A=A^*$.Hence $KerA=KerA^*$. Since from above dimKerA is finite, it easily follows that $dimKerA^*$ is also finite.
(iii) From *definition* 2.9 we see that A is a Fredholm operator with index zero.

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