GLOBAL JOURNAL OF MATHEMATICAL SCIENCES VOL. 3, NO.1, 2004: 23 - 26
COPYRIGHT(C) BACHUDO SCIENCE CO. LTD. PRINTED IN NIGERIA. ISSN 1596-6208

SOME DEFICIENCIES IN USING MOMENT GENERATING FUNCTIONS

ABRAHAM OKOLO

(Received 15 April 2003; Revision accepted 6 June 2003)

ABSTRACT

Two deficiencies in using moment-generating functions are given and illustrated with examples. Many distributions do not have moment generating functions, but every distribution has a unique characteristic function. The use of characteristic functions is preferred to moment-generating functions.

KEY WORDS: Moment - generating functions, Characteristic functions

INTRODUCTION

The moments of a density function play an important role in theoretical and applied statistics. In fact, in some cases, if all the moments are known, the density can be determined (Mood, Graybill and Boes, 1974, p.78). It has been pointed out in Hogg and Craig (1978) that "not every distribution has a moment-generating function" (MGF). However, "it is difficult to overemphasize the importance of a moment-generating function when it does exist. This importance stems from the fact that the moment-generating function is unique and completely determines the distribution of the random variable" (Hogg and Craig, 1978, p.50).

According to Hogg and Craig (1978), Niels and Buch (1987), and Knight (2000), so many distributions do not have moment-generating functions, but every distribution has a unique characteristic function (CF); and to each characteristic function there corresponds a unique distribution of probability. Consequent upon this, in a more advanced statistics course, the use of CF's is preferable to MGF's.

The reason that the MGF is a deficient tool in comparison to the CF is that the domain of a MGF depends on the distribution, while the domain of all CF's is the same - the real line. If the domain of the MGF is an open interval containing zero, then the corresponding CF is analytic in some strip containing the real line; in such a case the MGF can be used without difficulty, with some additional restrictions in limit theorems (Kotz and Johnson, 1982, p.420).

In this paper, two deficiencies of the MGF are given and exemplified. They are as follows:

Case 1. A random variable can have no moments at all, but it may have a MGF.
Case 2. A random variable can have some or all moments and a MGF, but the MGF does not generate the moments.

ILLUSTRATIVE EXAMPLES

Example 1. Let x have values n = 1, 2, ..., with probabilities $P(x = n) = 1/[n(n+1)]$. The expectation E(x) of this random variable does not exist because

$$E(x) = \sum_{n=1}^{\infty} n\, P(x = n) = \sum_{n=1}^{\infty} n \, . \, 1/[n(n+1)]$$

$$= \sum_{n=1}^{\infty} 1/(n+1),$$

is a series which is known (Dolciani et. al., 1980) to be divergent, therefore, no moments exist.

But the MGF, M(t), does exist for t 0, where

ABRAHAM OKOLO, Department of Statistics and Operations Research Federal University of Technology, Yola, Yola, Nigeria.

$$(\star) \qquad M(t) = \begin{cases} 1, & t = 0 \\ 1 + (e^{-t} - 1)\,\mathrm{Log}_e(1 - e^t), & t < 0 \\ \text{does not exist}, & t > 0 \end{cases}$$

This can be established as follows:

$$M(t) = E(e^{tx}) = \sum_{n=1}^{\infty} e^{nt} \,.\, 1/[n(n+1)]$$

$$= \sum_{n=1}^{\infty} (e^t)^n \,.\, 1/[n(n+1)].$$

Substituting ~ = e ^{t}, the probability generating function (PGF) of x is obtained:

$$M(t) = h(Z) = \sum_{n=1}^{\infty} Z^n \,.\, 1/[n(n+1)],$$

which is a power series with radius of convergence equal to one (Dolciani et. al., 1980). The sum of this series is

$$h(Z) = \sum_{n=1}^{\infty} Z^n \,.\, 1/[n(n+1)] = \sum_{n=1}^{\infty} Z^n \,.\, \{(1/n) - [1/(n+1)]\}$$

$$= \sum_{n=1}^{\infty} \frac{Z^n}{n} - \sum_{n=1}^{\infty} \frac{Z^n}{n+1}$$

$$= \sum_{n=1}^{\infty} \frac{Z^n}{n} - \left(\sum_{n=1}^{\infty} \frac{Z^n}{n+1} + \frac{Z^0}{1} - 1 \right)$$

$$= \sum_{n=1}^{\infty} \frac{Z^n}{n} - \sum_{n=0}^{\infty} \frac{Z^n}{n+1} + 1$$

$$= \sum_{n=1}^{\infty} \frac{Z^n}{n} - (1/Z) \sum_{n=0}^{\infty} \frac{Z^{n+1}}{n+1} + 1$$

$$= 1 + (Z^{-1} - 1)\,\mathrm{Log}_e(1 - Z),\ 0 < |Z| < 1$$

which yields ($\star$) for t < 0.

Example 2. Let x have values $2^0, 2^1, \ldots, 2^n, \ldots$ with probabilities

$$P(x = 2^n) = e^{-1}/n!\,,\ n = 0, 1, 2, \ldots .$$

All the moments of x exist because

$$E(x^r) = \sum_{n=0}^{\infty} (2^n)^r . P(x = 2^n) = \sum_{n=0}^{\infty} (2^r)^n e^{-1}/n!$$

$$= \exp(2^r - 1),\ r = 1, 2, \dots .$$

The MGF also exists; this can be obtained as follows:

$$M(t) = \sum_{n=0}^{\infty} \exp(t . 2^n) . P(x = 2^n) = \sum_{n=0}^{\infty} \exp(2^n t) . (e^{-1}/n!).$$

Denoting = e^t, the probability generating function of x is obtained:

$$M(t) = h(Z) = \sum_{n=0}(Z)^{2^n} . (\frac{e^{-1}}{n!}).$$

This series converges for . 1 and diverges for > 1 (Dolciani et. al., 1980, p.107). Thus the series in

$$M(t) = h(e^t) = \sum_{n=0}^{\infty} \exp(2^n t). (e^{-1}/n!)$$

converges for t 0 and diverges for t > 0, and M (t) is defined for t 0 only. This way M(t) cannot be differentiated at t = 0 or expanded into the Maclaurin formula. Hence, all the moments
exist here and the MGF exists as well, but it does not generate the moments.

Example 3. Let U and V be stochastically independent chi-square variables with n_1 and n_2

degrees of freedom, respectively. Then the continuous random variable $x = \frac{U/n_1}{V/n_2}$ has an

F-distribution with degrees of freedom n_1 and n_2 (Hogg and Craig, 1978, p.146)
The rth moment of x (for $n_2 > 2r$) will be:

$$E(x^r) = E\left[\left(\frac{U/n_1}{V/n_2}\right)^r\right] = \left[\frac{n_2}{n_1}\right]^r E(U^r)E(V^{-r}) = \left[\frac{n_2}{n_1}\right]^r \frac{(n_1/2)^r}{(n_2/2-1)^r}.$$

Kotz and Johnson (1983) have documented that, if $n_2 \le 2r$, the rth moment is infinite, therefore no moments exist. In fact, the F-distribution with n_1 and n_2 degrees of freedom, for n_2 = 1 or 2 is one of the distributions described in case 1.

Example 4. Let x be a continuous random variable, and let a new random variable Y be defined as Y = Log_e x. It has been reported in Mood, Graybill and Boes (1974) that if Y has a normal distribution with mean μ and variance σ^2, then x has a lognormal distribution with probability density function

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(\log_e x - \mu)^2\right) I_{(0,\infty)}(x).$$

All the moments of x exist because

$$E(x^r) = \int_0^\infty x^r f(x)\, dx = \exp\left(r\mu + \frac{r\sigma^2}{2}\right).$$

However, Kotz and Johnson (1985) have reported that the MGF for the lognormal distribution does not exist. Thus all the moments of the lognormal distribution exist, but these are not generated by the MGF. Accordingly, the lognormal distribution is one of the distributions described in case 2.

CONCLUSION

The deficiencies mentioned apply to both discrete and continuous random variables as illustrated by the examples on this paper.

REFERENCES

Dolciani, M. P., Beckenbach, E. F., Donnelly, A. J., Jurgensen, R. C. and Wooton, W., 1980. Modern Introductory Analysis. Houghton Mifflin Company, Boston, 210pp.

Hogg, C. V. and Craig, A. T., 1978. Introduction to Mathematical Statistics, Macmillan Publishing Co., Inc., New York, 438pp.

Knight, K., 2000. Mathematical Statistics. CRC Press, London, 420pp.

Kotz, S. and Johnson, N. L., 1982. Encyclopedia of Statistical Sciences. John Wiley and Sons, Inc., New York, 480pp.

Kotz, S. and Johnson, N. L., 1983. Encyclopedia of Statistical Sciences. John Wiley and Sons, Inc., New York, 722pp.

Kotz, S. and Johnson, N. L., 1985. Encyclopedia of Statistical Sciences. John Wiley and Sons, Inc., New York, 741pp.

Mood, A. M., Graybill, F. A. and Boes, D. C., 1974. Introduction to the Theory of Statistics. McGraw-Hill Kogakusha, Ltd., Tokyo, 564pp.

Niels, A. and Buch, K. R., 1987. Introduction to the Theory of Probability and Statistics. John Wiley and Sons, Inc., New York, 587pp.