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EXISTENCE AND UNIQUENESS OF SOLUTION OF FIRST ORDER QUASILINEAR HYPERBOLIC SYSTEM IN TWO-INDEPEDENDENT VARIABLES USING FIXED POINT THEOREMS

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ABSTRACT

Let $A(t,x,u)u_t + B(t,x,u)u_x = C(t,x,u)$ be a strictly hyperbolic $n \times n$ system with $u(0,x) = \phi(x)$ its initial data. Using the relative compactibility of the domain of dependence of solution, the contraction mapping principle and Schauder fixed point theorem, the existence and uniqueness of the solution to the Cauchy problem are established.

Key Words: Existence, Uniqueness, Strict Hyperbolicity, Compact and Contraction Operator.

1 INTRODUCTION

The existence and uniqueness of solution to the first - order hyperbolic system in two independent variables had been studied by some authors e.g. T. J. Langan (1976) for Semi - linear hyperbolic system, Courant R. (1961), Lee - Da Tsin and Yu Wen-Tzk (1981) for Quasi - linear hyperbolic systems. For the quasi - linear case, the characteristics of the system can only be determined when the solution to the system is known. It is accepted that smooth solution exists locally at the point of consideration and this solution has been shown to be the same for any of the types Courant R. (1961), Jeffrey (1976), Lee Da - Tsin (1981).
Nevertheless, the information about the solution is still yet incomplete. We can by-pass the determination of the solution before we consider the existence and uniqueness of solution by the use of function space theory.

Actually, the existence or construction of a solution to a differential equation, be it linear or non - linear is often reduced to the existence or location of a fixed point of an operator defined on a subset of the space specified as the domain of dependence for such solution. The fixed point so obtained is taken to be the desired solution or a representation of the solution to the given Cauchy problem provided that all the partial derivatives of the solution with respect to the independent variables exist.

In order to make use of any of the fixed point theorems, the space of functions and the type of operator must be clearly defined. We proceed as follows: The system of quasi - linear hyperbolic partial differential equations is reduced to a system of integral equations in a domain 'Ω' on which the solution at a given point is defined. Some relevant definitions and fixed point theorems are stated. Some lemmas are given and proved for clarity. Then we apply the theorems to the system of integral equations to prove that the solution of the original problem exists and is unique.

2 THE SYSTEM OF EQUATIONS AND REDUCTION TO SYSTEM OF INTEGRAL EQUATIONS

Consider the quasi-linear hyperbolic system of the form

$$A(t,x,u)u_t + B(t,x,u)u_x = C(t,x,u) \tag{2.1}$$

with

$$u(0,x) = \phi(x) \tag{2.2}$$

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where $u(t,x)$ is a column vector with components $(u_1, u_2, \dots, u_n)$, $A(t,x,u)$ and $B(t,x,u)$ are $n \times n$ coefficient matrices. All possess continuous partial derivatives in a domain to be specified. The $n \times n$ matrix $A(t,x,u)$ is assumed to be symmetric and non-singular. $u(0,x)$ is called the initial data. Equation (2.1) can be written in the form

$$u_t + M(t,x,u)u_x = d(t,x,u) \tag{2.3}$$

where $M(t,x,u) = A^{-1}B$ and $d(t,x,u) = A^{-1}C(t,x,u]$.

The matrix M possesses n-distinct eigenvalues $\lambda_k(t,x,u)$ and n real characteristics curves. Each characteristic curve is determined from the ordinary differential equation

$$\frac{dx}{dt} = \frac{-\xi_t}{\xi_x} = \lambda_k(t,x,u) \quad k = 12, 3, \dots, n \tag{2.4}$$

Definition 2.1 The system (2.3) is hyperbolic if the matrix M is diagonalizable and possess real eigenvalues $\lambda_k(t,x,u)$ and $\bar{l}_k(t,x,u)$, $k = 1, 2, \dots, n$ (the corresponding linearly independent eigenvectors / eigenfunctions) so that M can be decomposed in the form

$$M = S\Lambda S^{-1}$$

where $\Lambda = diag(\lambda_1, \lambda_2, \dots, \lambda_n)$ and $S = (\bar{l}_1|\bar{l}_2|\dots|\bar{l}_n)$ is the matrix of the left eigenvectors. The matrix S is also called the modal matrix of M(Mikhailov (1978), Valli (1996).

Let the eigenvalues be ordered so that

$$\lambda_1 > \lambda_2 \dots, > \lambda_n \quad \text{and} \quad c_1,\ c_2,\ c_3, \dots c_n$$

denote the respective characteristic curves. Equation (2.3) can therefore be written in the form

$$u_t + S\Lambda S^{-1}u_x = d(t,x,u) \tag{2.5}$$

Multiplying equation (2.5) by S^{-1} on the left and substituting $V = S^{-1}u$, we then have

$$V_t + \Lambda V_x = S^{-1}d - S_t^{-1}SV - S_x^{-1}SV$$

which can compactly be written as

$$V_t + \Lambda V_x = F(t,x,V). \tag{2.6}$$

Hence for the k-th component using (2.6), the Cauchy problem becomes,

$$\frac{\partial V_k}{\partial t} + \lambda_k \frac{\partial V_k}{\partial x} = F_k(t,x,V_k) \tag{2.7}$$

with

$$V_k(0,x) = S^{-1}\phi_k(x) \tag{2.8}$$

Let $P(\xi,\eta)$ be any point on the $t-x$ plane at which we wish to find the solution. The point $P(\xi,\eta)$ is therefore a point of intersection of all the characteristic curves having their value P_k at the point of intersection with the initial curve. Along the characteristic curves at $t = \tau$,

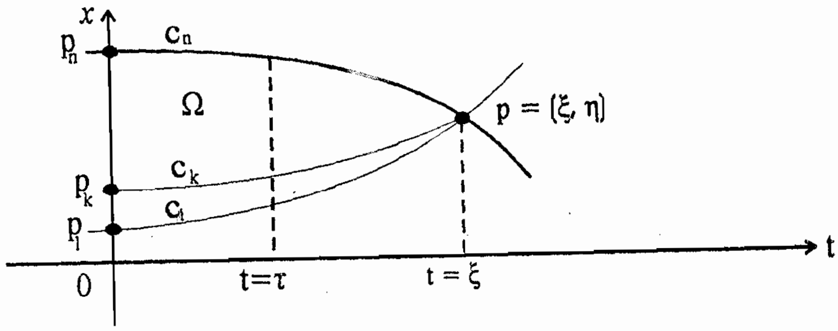
$$X_k(t) = Y_k(\tau,\xi,\eta) \tag{2.9}$$

The solution to equations (2.7) and (2.8) is obtained by integrating with respect to t for $0 \le t \le \xi$ and replacing X with equation (2.9). We thus have

$$V_k(P) = V_k(P_k) + \int_0^{\xi} F_k(t, Y_k(t,\xi,\eta),\ V_k(t, Y_k(t,\xi,\eta))dt \tag{2.10}$$

Equation (2.10) is the integral equation representation of the solution of the Cauchy problem (2.7) and (2.8). It shows that the value of the solution at $P(\xi,\eta)$ depends upon the value of

F_k and V_k within a domain Ω bounded by the highest and lowest characteristic curves c_n and c_1 and the line P_1P_n (see Fig. 1).



The region Ω is called the domain of dependence of the solution at the point $P(\xi, \eta)$ to the Cauchy problem given by the equations (2.1) and (2.2). When V_k is determined we use equation $SV_k = U_k$ to obtain the component form of the required solution.

3. FIXED POINT THEOREMS AND APPLICATION TO (2.10)

Consider the bounded region Ω, a subset of $\mathbb{R} \times \mathbb{R}^n$. Let the set $C(\Omega)$ denote the space of all continuous functions in Ω.

Definition 3.1 In $C(\Omega)$ the subsets $C^k(\Omega)$ for $k = 1, 2, \ldots,$ consists of the functions whose partial derivatives up to order k are continuous in Ω with norm defined by

$$\|f\|_{c(\Omega)} = \sum_{|\alpha| \leq k} \max_{x \in \Omega} |D^\alpha f(x)|. \tag{3.1}$$

where Ω is a compact set in $R \times R^n$ and $k \in \mathbb{Z}$. The subset $C^k(\Omega)$ is a linear space. We see that convergence in the norm given by (3.1) means uniform convergence in Ω of the functions and their partial derivatives up to order k. This space is called a Banach Space with norm (3.1) according to Mikhailov [(1978, p. 102].

Definition 3.2 A subset H of $C(\Omega)$ is said to be equi-continuous if, for any $\epsilon > 0$, there exists $\delta > 0$ such that $|p_1 - p_2| < \delta$ implies that $|F(p_1) - F(p_2)| < \varepsilon \ \forall F \in H$ and $\forall p_1, p_2 \in \Omega$. Moreover it will be said to be uniformly bounded if there exists a constant $C^* > 0$ such that

$$|F|_\infty \leq C^* \quad \forall F \in H; \quad |F|_\infty := \sup\{\|F(P)\| : P \in \Omega)\}(Chidume(1995),\ p.161)$$

We state the following

Lemma 3.1: Assume that H satisfies the definition 3.2 and $F_k(t, x, v_k)$ be continuous in H with respect to v_k so that

(i) $\exists M \geq 0 \quad \ni |F_k(t, x, v_k)| \leq M, \ \forall k$

(ii) $\exists L \geq 0 \quad \ni |F_k(t, x, v_k^2) - F_k(t, x, v_k')| \leq L|v_k^2 - v_k^1|$ holds for any two points (t, x, v_k^2) and (t, x, v_k^1) in H. Then H is relatively compact.

Proof: Let $F_k(t, x, v_k)$ be arbitrary and continous in H. Consider two points in H, one is fixed, say (t, x, v_k^*) and (t, x, v_k) is arbitrary. Then

$$\begin{aligned} |F_k(t, x, v_k)| &\leq |F_k(t, x, v_k) - F_k(t, x, v_k^*)| + |F_k(t, x, v_k^*)| \\ &\leq L|v_k - v_k^*| + |F_k(t, x, v_k)| \\ &\leq L\delta + M = C^* \end{aligned}$$

Define

$$|F_k|_\infty = \sup_{p\in\Omega} |F_k(t,x,v_k)| \leq C^*$$

Hence H is uniformly bounded.
Furthermore, if we choose $\delta = \frac{\epsilon}{L+1}$ for a given $\epsilon > 0$

we obtain $|F_k(t,x,v_k^2) - F_k(t,x,v_k^l)| < \epsilon$ for any two points in H. Hence, by Arzela-Ascoli theorem, H is relatively compact.

Any set of elements in a Banach space is contained in some closed convex set. There exists a smallest closed convex set $\overline{\Omega}$ containing Ω called the convex hull of Ω.

By lemma 3.1 the space Ω is relatively compact and so it is a subset of a Banach Space, in that Ω contain its convex hull i.e. $\Omega = \overline{\Omega}$. This implies that every infinite sequence of elements in Ω contains a subsequence converging to an element in that same space.

Definition 3.2 Let X and Y be Banach Spaces and let $T: D(T) \subseteq X \to Y$ be an operator. T is called a compact operator if and only if (i) T is continuous and (ii) T maps bounded sets into relatively compact sets.

Lemma 3.2 Let $F_k \in C'(\Omega)$ and Ω is a non-empty relatively compact subset of a Banach Space. Then $T: \Omega \to \Omega$ defined by

$$TV_k = T(t,x,v_k) = \int_0^\xi F_k(t,x,v_k)dt \tag{3.2}$$

is a compact operator and contracting in the maximum norm.

Proof: Let μ be the upper bound for the F_k and its partial derivatives with respect to the independent variables. For sufficiently small strip 'h' in the neighbourhood of the x-axis we have

$$|TV_k^n - TV_k^{n-1}| = |\int_0^\xi (F_k(t,x,v_k^n) - F_k(t,x,v_k^{n-1}))dt|$$

so that

$$|TV_k^n - TV_k^{n-1}| \leq \int_0^\xi |F_k(t,x,v_k^n) - F(t,x,v_k^{n-1})|dt \tag{3.3}$$

Since $F_k(t,x,v_k) \in C'(\Omega)$, then we apply the mean value theorem to the right-hand side of (3.3) to get

$$|TV_k^n - TV_k^{n-1}| \leq \int_0^\xi |F_k'(t,x,v_k)||v_k^n - v_k^{n-1}|dt$$

and by using the maximum norm we get eventually

$$\|TV_k^n - TV_k^{n-1}\| \leq h\mu\|v_k^n - v_k^{n-1}\|, \tag{3.4}$$

where h is sufficiently small that $h\mu < 1$ and the operator therefore is both continuous and contracting, implying that $\|T\| \leq 1$.
Let $\{V_k^n\}_{n=1}^\infty$ be a sequence in Ω which converges uniformly to $U \in \Omega$. It follows from (3.4) that

$$\begin{aligned}\|TV_k^n - TU\|_{c(\Omega)} &= \max|\int_0^\xi (F_k(t,x,v_k^n)) - F(t,x,u)dt| \\ &\leq \|V_k^n - U\| \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

Therefore by the uniform continuity of F_k, T is compact.

Contraction Mapping Principle

Let Ω be a compact set and let $T: \Omega \to \Omega$ be a contraction map. Then T has a unique fixed point in Ω. Moreover, for any $U_0 \in \Omega$, the sequence $\{T_n U_0\}_{n=1}^\infty$ converges to the fixed point (Martin (1976), p. 114).

Schauder Fixed Point Theorem

Let Ω be a non-empty, closed, bounded, convex subset of a Banach Space X and suppose $T: \Omega \to \Omega$ is a compact operator. Then T has a fixed point (Zeidler (1986) p. 57).

Using the above fixed point theorems we now express the main result of this paper in the following theorem.

Theorem 3.1 (Existence and Uniqueness)

Let Ω (as in fig. 1) which is the region bounded by the lower and upper characteristics and the axis $t = 0$, be a compact subset of a Banach Space and let the system (2.1) and (2.2) be strictly hyperbolic in Ω. Under the assumption of lemma (3.2) there exists a global solution to the Cauchy problem (2.1) and (2.2) and this solution is unique.

Proof: For arbitrary fixed point $P(\xi, \eta) \in \Omega$, and strict hyperbolicity of (2.1) in Ω, the solution at $P(\xi, \eta)$ with representation as in (2.10) we denote by U_0. We define a sequence $\{U_n\} \in \Omega$ by $U_n = T_n U_0, \quad n = 1, 2, 3, \ldots$. Assume that $\exists n, m$ with $n > m$ such that

$$\|U_n - U_m\| \leq \sum_{k=m+1}^{n} \|U_k - U_{k-1}\|$$

by the method of successive approximations.

Then

$$\begin{aligned} \|U_n - U_m\| &\leq \sum_{k=m+1}^{n} \|T_{k-1}U_1 - T_{k-1}U_0\| \\ &\leq \sum_{k=m+1}^{n} \mu^{k-1}\|U_1 - U_0\| \\ &\leq \frac{\mu^m}{1-\mu}\|U_1 - U_0\|. \end{aligned} \tag{3.5}$$

Since T is compact and contracting then by the contraction mapping principle we have from (3.5)

$$\lim_{m \to \infty} \frac{\mu^m}{1-\mu}\|U_1 - U_0\| \to 0.$$

Hence $\lim_{n,m \to \infty} \|U_n - U_m\| \to 0$

Thus $\{U_n\}$ is a Cauchy sequence. Since Ω is compact, U_n converges to an element U_0 in Ω i.e $U_n \to U_0$. Then by the uniform mapping principle, $TU_n \to TU_0$ and we conclude by Schauder fixed point theorem that there is a fixed point U_0 such that $TU_0 = U_0$. Hence, the solution to the Cauchy problem (2.1) and (2.2) exists.

We now show that the solution U_0 is unique.

Suppose that $\exists V_0 \ni TV_0 = V_0$, another fixed point, then

$$\|U_0 - V_0\| = \|TU_0 - TV_0\| \leq \mu\|U_0 - V_0\| \Rightarrow \mu \geq 1, \tag{3.6}$$

and $\frac{\|TU_0 - TV_0\|}{\|U_0 - V_0\|} \leq \mu$. Therefore

$$\frac{\|T(U_0 - V_0)\|}{\|U_0 - V_0\|} \leq \frac{\|T\|\|U_0 - V_0\|}{\|U_0 - V_0\|} \leq \mu \leq 1, \tag{3.7}$$

since $\|T\| \leq 1$. The inequalities (3.6) and (3.7) lead to a contradiction and can only hold if $U_0 = V_0$. Hence U_0 is unique.

CONCLUSION

Given a Cauchy problem, of a the type 2.1 and 2.2 where the system is strictly hyperbolic. If it is known that the region of the domain of dependence is compact, then by the application of some fixed point theorems, the solution to the problem is shown to exists and is unique.

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