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FIXED POINT THEOREMS FOR ASYMPTOTICALLY PSEUDOCONTRACTIVE MAPS IN CERTAIN BANACH SPACES

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ABSTRACT

Let E be a real q-uniformly smooth Banach space (for example, L_p or lp spaces ,$1<p<\infty$) and let $T: E\to E$ be asymptotically pseudocontractive mapping with nonempty fixed point set F(T) and a real sequence $\{k_n\}_{n=1}^{\infty}$. Let $\{x_n\}_{n\geq1}$ be the sequence generated from an arbitrary x_1 in E by $x_{n+1}=(1-\alpha_n)x_n+\alpha_n T^n x_n$, $n\geq1$. If the range of T is bounded and there exists a strictly increasing function $\phi:[o,\infty)\to[o,\infty)$ with $\Phi(0)=0$ such that $\langle T^n x_n-p, j(x_n-p)\rangle \leq k_n\|x_n-p\|^2-\Phi(\|x_n-p\|)$, $p\in F(T)$, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to p, provided that $\{k_n\}$ and $\{\alpha_n\}$ satisfy certain properties. This result compliments the results of Chang, Park and Cho (2000), by dropping the Lipschitz condition on T.

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1. INTRODUCTION

Let E be an arbitrary real Banach space and let J_q ($q>1$) denote the *generalized duality mapping* from E into 2^{E^*} given by $J_q(x)=\{f\in E^*: \langle x,f\rangle=\|x\|^q \text{ and } \|f\|=\|x\|^{q-1}\}$, where E* denotes the *dual space of E* and $\langle.,.\rangle$ denotes the *genaralised duality pairing*. In particular, J_2 is called the *normalized duality mapping* and is usually denoted by J. It is well known (see for example Xu (1991)) that $J_q(x)=\|x\|_{q-2}J(x)$ if $x\neq0$, and that if E is uniformly smooth then Jq is single-valued and uniformly continuous on bounded sets. In the sequel we shall denote the single-valued generalized duality mapping by j_q.

Let E be a real Banach space. The *modulus of smoothness* of E is the function $\rho_E:[0,\infty)\to[0,\infty)$ defined by

$$\rho_E(\tau)=\sup\{½(\|x+y\|+\|x-y\|)=1:\|x\|\leq1,\ \|y\|\leq\tau\}.$$

E is *uniformly smooth* if and only if $\lim(\rho_E(\tau)/\tau)=0$ as $\tau\to0$. Hilbert spaces, Lp (or lp) spaces, with $1<p<\infty$,and the Sobolev spaces, W^p_m, m a positive integer , and $1<p<\infty$, are q-uniformly smooth. Hilbert spaces are 2-uniformly smooth while

$$\text{Lp (or lp) or } W^p_m \text{ is } \begin{cases} \text{p-uniformly smooth if } 1<p\leq2, \\ \text{2-uniformly smooth if } p\geq2. \end{cases}$$

The following theorem gives a geometric characterization of q-uniformly smooth Banach spaces :

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Theorem HKX (Xu (1991), P.1130): Let $q > 1$ and let E be a real Banach space. Then the following statements are equivalent:

1. E is q-uniformly smooth

2. There exists a constant $c_q > 0$ such that for all $x,y \in E$

$$||x+y||^q \leq ||x||^q + q\langle y, j_q(x)\rangle + c_q ||y||^q \quad (1)$$

3. There exists a constant d_q such that for all $x, y \in E$, and $t \in [0, 1]$

$$||(1-t)x + ty||^q \geq (1-t) ||x||^q + t||y||^q - w_q(t)\, d_q ||x-y||^q, \quad (2)$$

where $w_q(t) = t^q (1-t) + t(1-t)^q$.

Let K be a nonempty subset of an arbitrary real Banach space E. A mapping $T:K \to K$ is called *asymptotically pseudocontractive* with sequence $\{k_n\} \subseteq [1,\infty)$, $\lim_{n\to\infty} k_n = 1$ (see Schu (1991), Chang et al (2000), Osilike and Igbokwe (2002)) if for all $x,y \in K$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle T^n x - T^n y, j(x-y)\rangle \leq k_n ||x-y||^2 \quad (3)$$

for all $n \in N$. As shown in Schu (1991), this class of mapping is more general than the important class of *asymptotically nonexpansive mapping* (ie mappings $T:K \to K$ such that $||T^n x - T^n y|| \leq k_n ||x-y||$, for all $x, y \in K$, for some sequence $\{k_n\}$ with $\lim k_n = 1$, $n \to \infty$ and for all $n \in N$) introduced by Goebel and Kirk (1972). T is called *uniformly L-Lipschitzian* if $||T^n x - T^n y|| \leq L\, || x-y ||$, for all $x, y \in K$, $n \in N$ and for some $L > 0$.

The class of asymptotically pseudocontractive maps was introduced by Schu (1991). In (1991) Schu also introduced the *modified Mann iteration method* $\{x_n\}$ generated from an arbitrary $x_1 \in K$ by

$$x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n T^n x_n,\ n \geq 1, \quad (4)$$

where $\{\alpha_n\}$ is a real sequence in [0, 1].

In 2000, Chang, Park and Cho proved the following theorem:

Theorem CPC (Chang et al (2000), Theorem 2.3): Let K be a nonempty closed convex subset of E. Let $T:K \to K$ be a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with a sequence $\{k_n\}$ in $[1,\infty)$ such that $\lim_{n\to\infty} k_n = 1$.

Let $\{\alpha_n\}$ be a sequence in [0, 1] satisfying the following conditions:

(i) $\alpha_n \to 0$ as $n \to \infty$ (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Let $\{x_n\}$ be the modified Mann iterative sequence defined by (4). If $F(T) = \{x \in K : Tx = x\} \neq \phi$, and if for any given $p \in F(T)$, there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x_{n+1} - p, j(x_{n+1} - p)\rangle \leq k_n \|x_{n+1} - p\|^2 - \phi(\|x_{n+1} - p\|) \quad (5)$$

for all $n \geq 1$, where $j(x_{n+1} - p) \in J(x_{n+1} - p)$ is such that $\langle T^n x_{n+1} - T^n p, j(x_{n+1} - p)\rangle \leq k_n \|x_{n+1} - p\|^2$, $\forall\ n \geq 1$, then the sequence $\{x_n\}$ converges strongly to the fixed point p of T.

The assumption that T is uniformly L-Lipschtizian appears restrictive since it is not satisfied by many continuous maps.

It is our purpose in this paper to enlarge the class of operators satisfying Theorem CPC by dropping the uniform L-Lipschitizian condition. No continuity assumption will be made on the operator T. Furthermore, we shall extend the result to the more general modified Mann iteration methods with errors in the senses of Liu (1995) and Xu (1998). We shall prove our result in q-uniformly smooth spaces. Our method of proof is short. Our results complement those of Chang, Park and Cho (2000) and a very recent result of Osilike and Igbokwe (2002).

2. MAIN RESULTS

In the sequel, we shall require the following Lemma:

Lemma 2.1: Let $\{a_n\}$, $\{b_n\}$ and $\{t_n\}$ be nonnegative real sequences such that

$\sum t_n = \infty$, $t_n \to 0$ as $n \to \infty$, $b_n = 0(t_n)$ and satisfy

$$a^q_{n+1} \leq a^q_n - t_n \phi(a_n) + b_n,\ n \geq 1 \quad (6)$$

where $\phi : [0, \infty) \to [0, \infty)$ is a strictly increasing function with $\phi(0) = 0$. Then $a_n \to 0$ as $n \to \infty$.

Proof: The prove of the lemma follows from the following two claims:

Claim 1: $\liminf a_n = 0$.

Assume the contrary and let $\liminf a_n = \delta > 0$. Then there exists a positive integer N_0 such that $\forall\ n \geq N_0$, $a_n > \delta/2$. Also $b_n = 0(t_n)$ implies that $\exists\ N_1 > 0$ such that $b_n/t_n \leq ½\,\phi(\delta/2)$ (ie $b_n \leq ½\,\phi(\delta/2)\,t_n$), $\forall\ n \geq 1$. Define N $\max\{N_0, N_1\}$. Then $\forall\ n \geq N$, (6) implies

$$a^q_{n+1} \leq a^q_n - t_n \phi(\delta/2) + ½\,\phi(\delta/2)\,t_n$$

$$\leq a^q_n - ½\,\phi(\delta/2)\,t_n,$$

so that

$$½\,\phi(\delta/2)\,t_n \leq a^q_n - a^q_{n+1},$$

Therefore, $\forall\ n \geq N$,

$$\tfrac{1}{2}\phi(\delta/2)\sum_{j=N}^{n} t_j \leq a^q_n.$$

Hence, $\Sigma t_n < \infty$, contradicting $\Sigma t_n = \infty$. Thus $\liminf a_n = 0$. Therefore, there exists a subsequence $\{a_{nj}\}$ of $\{a_n\}$ such that $a_{nj} \to 0$ as $j \to \infty$. Thus given any $\varepsilon > 0$ there exists an integer $j_0 > 0$ such that $\forall j \geq j_0$ $a_{nj} < \varepsilon/2$. Also, $\exists N_2 > 0$ such that $\forall n \geq N_2$, $b_n < \tfrac{1}{2}\phi(\varepsilon/2)\, t_n$.
Denote n_{j^*} by n^* and set $n^* \geq \max\{nj^*, N_2\}$.

Claim 2: $a_{n^*+k} < \varepsilon/2,\ \forall\ k \geq 0.$

For $k = 0$, the claim is clearly true. We assume that the claim is true for some $k > 0$. Suppose the claim is false for k+1. Then $a_{n^*+k+1} \geq \varepsilon/2$. But from (6)

$$\begin{aligned}\varepsilon^q/2^q \leq a^q_{n^*+k+1} &\leq a^q_{n^*+k} - t_{n^*+k}\,\phi(a_{n^*+k+1}) + b_{n^*+k}\\ &< (\varepsilon/2)^q - t_{n^*+k}\,\phi(\varepsilon/2) + \tfrac{1}{2} t_{n^*+k}\,\phi(\varepsilon/2)\\ &= \varepsilon^q/2^q - \tfrac{1}{2}\phi(\varepsilon/2)\, t_{n^*+k} < \varepsilon^q/2^q,\end{aligned}$$

which is a contradiction. Claim 2 is therefore proved. Since $\varepsilon > 0$ is arbitrary, this implies that $a_n \to 0$ as $n \to \infty$, completing the proof of the lemma.

Theorem 2.2: Let E be a real q-uniformly smooth Banach space and let $T: E \to E$ be an asymptotically pseudocontractive mapping with a nonempty fixed point set F(T) and a real sequence $\{k_n\}$ in $[1, \infty)$ such that $\lim k_n = 1$, $n \to \infty$. Let $\{\alpha_n\}$ be a real sequence in [0, 1] satisfying (i) $\lim \alpha_n = 0$, $n \to \infty$ and (ii) $\Sigma \alpha_n = \infty$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in E$ by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1 \tag{7}$$

If the range of T is bounded and if for some $p \in F(T)$ there exists a strictly increasing function $\phi: [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x_n - p, j(x_n - p)\rangle \leq k_n \|x_n - p\|^2 - \phi(\|x_n - p\|), \tag{8}$$

then $\{x_n\}$ converges strongly to p.

Proof: Set $M \doteq \sup_{n\geq 1} \|T^n x_n - p\|$. Using (4), (1) and (5) we obtain:

$$\begin{aligned}\|x_{n+1} - p\|^q &= \|(1-\alpha_n)(x_n - p) + \alpha_n(T^n x_n - p)\|^q\\ &\leq (1-\alpha_n)^q \|x_n - p\|^q + q(1-\alpha_n)\alpha_n \langle T^n x_n - p, j_q(x_n - p)\rangle\\ &\quad + c_q \alpha^q_n \|T^n x_n - p\|^q\\ &\leq [(1-\alpha_n)^q + q(1-\alpha_n)\alpha_n k_n]\ \|x_n - p\|^q\\ &\quad - q(1-\alpha_n)\alpha_n \phi(\|x_n - p\|) + c_q \alpha^q_n M^q\end{aligned}$$

$$\leq [1 - q\alpha_n + \alpha^q_n + q(1-\alpha_n)\alpha_n k_n]\ ||x_n - p||$$
$$- q(1-\alpha_n)\alpha_n \phi(||x_n - p||) + c_q \alpha^q_n M^q$$
$$\leq [1 + q\alpha_n(x_n - 1)]\ ||x_n - p||^q - q(1-\alpha_n)\alpha_n \phi(||x_n - p||)$$
$$+ c_q \alpha^q_n M^q \qquad (9)$$

Let $D \doteq \sup\{||T^n x_n - p||, n \geq 1\} + ||x_1 - p||$. Then by simple induction $||x_n - p|| \leq D$, $\forall\ n \geq 1$.

It follows from (9) that

$$||x_{n+1} - p||^q \leq ||x_n - p||^q - q(1-\alpha_n)\alpha_n \phi(||x_n - p||) + b_n, \quad \forall\ n \geq 1 \qquad (10)$$

where

$$b_n = c_q \alpha^q_n M^q + q\alpha_n(k_n - 1) D^q.$$

Now, if we set $a_n \doteq ||x_n - p||$, and $t_n \doteq q(1-\alpha_n)\alpha_n$, in (10) we obtain

$$a^q_{n+1} \leq a^q_n - t_n \phi(a_n) + b_n, \qquad \text{which is (6).}$$

Thus by lemma 2.1, $x_n \to p$ as $n \to \infty$, completing the proof of theorem 2.2.

Remark : Once convergence results have been proved for the original modified Mann iteration method (4), the extensions of the results to the modified iteration method with errors in the sense of Liu (1995) and Xu (1998) are usually straight forward on imposing the necessary conditions on the error term. Hence, we have the following results whose proofs are omitted because they follow by straight forward modifications of the proofs on the corresponding results for the original modified Mann iteration method (4).

Theorem 2.3: Let E, T and α_n be as in Theorem 2.2. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in E$ by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n + u_n, \quad n \geq 1$$

where $\Sigma ||u_n|| < \infty$. If the range of T is bounded and if for some $p \in F(T)$, there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x_n - p, j(x_n - p)\rangle \leq k_n ||x_n - p||^2 - \phi(||x_n - p||),$$

then $\{x_n\}$ converges strongly to p.

Theorem 2.4: Let E and T be as in theorem 2.2. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in E$ by

$$x_{n+1} = a_n x_n + b_n T^n x_n + c_n u_n, \quad n \geq 1, \qquad (11)$$

where (i) $a_n + b_n + c_n = 1$ (ii) $\Sigma b_n = \infty$, $\lim b_n = 0$ and (iii) $\Sigma c_n < \infty$.

If $\{u_n\}$ is a bounded sequence in E, the range of T is bounded and for some $p \in F(T)$ there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$\langle T^n x_n - p, j(x_n - p)\rangle \le k_n ||x_n - p||^2 - \phi(||x_n - p||)$,

then $\{x_n\}$ converges strongly to p.

Set $\alpha_n = b_n + c_n$ *in (11) to obtain*

$$x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n T^n x_n + c_n (u_n - T^n x_n), \quad \forall \quad n \ge 1.$$

The rest of the proof follows as in the proof of Theorem 2.2.

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